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by Bambang Suprihatin

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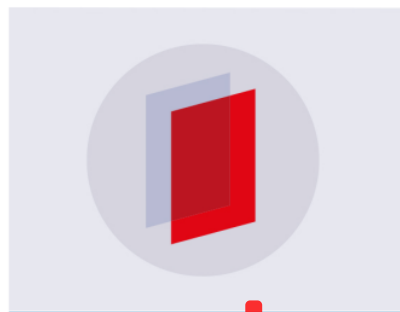
Limiting Distribution of the Bootstrap Parameter Estimator of a Second Order Autoregressive Model

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Abstract. This paper is the extension of our research about asymptotic distribution of the bootstrap parameter estimator for the AR(1) model. We investigate the asymptotic distribution of the bootstrap parameter estimator of a second order autoregressive AR(2) model by applying the delta method. The asymptotic distribution is the crucial property in inference of statistics. We conclude that the bootstrap parameter estimator of the AR(2) model asymptotically converges in distribution to the bivariate normal distribution.

1. Introduction

Let the stationary second order autoregressive AR(2) model:

$$X_t = \theta_1 X_{t-1} + \theta_2 X_{t-2} + \epsilon_t, \tag{1}$$

where ϵ_t is a white noise process with zero mean and constant variance σ^2. Suppose the vector $\widehat{\boldsymbol{\theta}} = (\widehat{\theta}_1, \widehat{\theta}_2)^T$ is the estimator of the parameter vector $\boldsymbol{\theta} = (\theta_1, \theta_2)^T$ of (1) and $\widehat{\boldsymbol{\theta}}^*$ be the bootstrap version of $\widehat{\boldsymbol{\theta}}$.

The consistency theories of parameter of autoregressive model have studied in [1,4,5], while for the bootstrap version of such topic, see *e.g.* [2, 6–9, 11]. For the accuracy of the bootstrapping method on autoregressive model studied in [3, 10]. The asymptotic result for the first order autoregressive or AR(1) model has been exhibited in [12]. We showed that the bootstrap parameter estimator for the AR(1) model has limiting that converges in distribution to the normal distribution. A good result of the bootstrap estimator is then applied to study the limiting distribution of $\widehat{\boldsymbol{\theta}}^*$. Section 2 reviews the asymptotic distribution of estimator of mean and autocovariance function for the autoregression model. Section 3 describes the bootstrap and delta method. Section 4 deals with the main result, *i.e.* the limiting distribution of $\widehat{\boldsymbol{\theta}}^*$. Section 5 briefly describes the conclusions of the paper.

2. Estimator of Mean and Autocovariance for the Autoregressive Model

Let we have the observed values $X_1, X_2, \ldots, X_n$ at hand taken from the stationary time series of an AR(2) model. Then consider the estimators as follows: $\widehat{\mu}_n = \overline{X}_n = \frac{1}{n} \sum_{t=1}^{n} X_t$, $\widehat{\gamma}_n(h) = \frac{1}{n} \sum_{t=1}^{n-h} (X_{t+h} - \overline{X}_n)(X_t - \overline{X}_n)$, and $\widehat{\rho}_n(h) = \widehat{\gamma}_n / \widehat{\gamma}_n(0)$ respectively. These three estimators have been showed that are consistent (see, *e.g.*, [4,14]). The following theorem describes the property of the estimator $\overline{X}_n$, is stated in [4].

Theorem 2.1 *If $\{X_t\}$ is stationary process with mean μ and autocovariance function $\gamma(\cdot)$, then as $n \to \infty$,*

$$Var(\overline{X}_n) = E(\overline{X}_n - \mu)^2 \to 0 \quad if \, \gamma(n) \to 0,$$

and

$$nE(\overline{X}_n - \mu)^2 \to \sum_{j=-\infty}^{\infty} \gamma(h) \quad if \sum_{j=-\infty}^{\infty} |\gamma(h)| < \infty.$$

It is not a loss of generality to assume that $\mu_X = 0$. Under some conditions (see, *e.g.,* [14]),

$$\widehat{\gamma}_n(h) = \frac{1}{n} \sum_{t=1}^{n-h} X_{t+h} X_t + O_p(1/n). \tag{2}$$

The asymptotic behaviour of the sequence $\sqrt{n}\left(\widehat{\gamma}_n(h) - \gamma_X(h)\right)$ depends only on $n^{-1} \sum_{t=1}^{n-h} X_{t+h} X_t$. Note that to study the behavior of (2) we can equivalently study the average

$$\widetilde{\gamma}_n(h) = \frac{1}{n} \sum_{t=1}^{n} X_{t+h} X_t. \tag{3}$$

Both (2) and (3) are unbiased estimators of $E(X_{t+h} X_t) = \gamma_X(h)$, under the condition that $\mu_X = 0$. Their asymptotic distribution then can be derived by applying a central limit theorem to the averages $\overline{Y}_n$ of the variables $Y_t = X_{t+h} X_t$. As in [14], the autocovariance function of the series Y_t can be written as

$$V_{h,h} = \kappa_4(\varepsilon)\gamma_X(h)^2 + \sum_g \gamma_X(g)^2 + \sum_g \gamma_X(g+h)\gamma_X(g-h), \tag{4}$$

where $\kappa_4(\varepsilon) = E(\varepsilon_1^4) - 3(E(\varepsilon_1^2))^2$, the fourth cumulant of ε_t. The following theorem is due to [14], gives the limiting distribution of $\sqrt{n}\left(\widehat{\gamma}_n(h) - \gamma_X(h)\right)$.

Theorem 2.2 *If $X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j}$ holds for an i.i.d. sequence ε_t with mean zero and $E(\varepsilon_t^4) < \infty$ and numbers ψ_j with $\sum_j |\psi_j| < \infty$, then*

$$\sqrt{n}\left(\widehat{\gamma}_n(h) - \gamma_X(h)\right) \to_d N(0, V_{h,h}).$$

3. Bootstrap and Delta Method
Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a population with common distribution F, and let $T(X_1, X_2, \ldots, X_n; F)$ be the specified random variable or statistic of interest, possibly depending upon the unknown distribution F. Let F_n denote the empirical distribution function of the random sample $X_1, X_2, \ldots, X_n$, *i.e.,* the distribution putting probability $1/n$ at each of the points $X_1, X_2, \ldots, X_n$. A bootstrap sample is defined to be a random sample of size n drawn from F_n, say $X^* = X_1^*, X_2^*, \ldots, X_n^*$. The bootstrap method is to approximate the distribution of $T(X_1, X_2, \ldots, X_n; F)$ under F by that of $T(X_1^*, X_2^*, \ldots, X_n^*; F_n)$ under F_n.

Let T be a functional defined as $T(X_1, X_2, \ldots, X_n; F) = \sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})$, where $\widehat{\boldsymbol{\theta}}$ is the estimator for the parameter $\boldsymbol{\theta}$ of a stationary time series AR(2) model. Then, the bootstrap version of T is $T(X_1^*, X_2^*, \ldots, X_n^*; F_n) = \sqrt{n}(\widehat{\boldsymbol{\theta}}^* - \widehat{\boldsymbol{\theta}})$, where $\widehat{\boldsymbol{\theta}}^*$ is a bootstrap version of $\widehat{\boldsymbol{\theta}}$ which is obtained by replacing the sample $X_1, X_2, \ldots, X_n$ by sample bootstrap $X_1^*, X_2^*, \ldots, X_n^*$. The residuals bootstrapping procedure for the time series data to obtain $X_1^*, X_2^*, \ldots, X_n^*$ was proposed in [7]. We should examine the distribution of $\sqrt{n}(\widehat{\boldsymbol{\theta}}^* - \widehat{\boldsymbol{\theta}})$ contrast to that of $\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})$. By bootstrapping, we estimate $P_F(\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \leq x)$ by $P_{F_n}(\sqrt{n}(\widehat{\boldsymbol{\theta}}^* - \widehat{\boldsymbol{\theta}}) \leq x)$. In this paper, we propose the delta method for investigate the limiting distribution.

The delta method is useful tool in order to deduce the limit law of $\phi(T_n) - \phi(\theta)$ from that of $T_n - \theta$, which has been proved through the following theorem, as stated in [13].

Theorem 3.1 *Let* ϕ : $\mathbf{D}_\phi \subset \mathbf{R}^k \to \mathbf{R}^m$ *be a map defined on a subset of* $\mathbf{R}^k$ *and differentiable at* θ. *Let* T_n *be random vector taking their values in the domain of* ϕ. *If* $r_n(T_n - \theta) \to_d T$ *for numbers* $r_n \to \infty$, *then* $r_n(\phi(T_n) - \phi(\theta)) \to_d \phi'_\theta(T)$. *Moreover,* $\left| r_n(\phi(T_n) - \phi(\theta)) - \phi'_\theta(r_n(T_n - \theta)) \right| \to_p 0$.

By Assuming that $\widehat{\theta}_n$ is a statistic, and ϕ is a given measurable function. The bootstrap version for the distribution of $\phi(\widehat{\theta}_n - \phi(\theta))$ is $\phi(\widehat{\theta}_n^* - \phi(\widehat{\theta}_n))$. The bootstrap method is consistent for estimating the distribution of the distribution of $\sqrt{n}(\phi(\widehat{\theta}_n) - \phi(\theta))$, as in the following theorem.

Theorem 3.2 *Let* ϕ : $\mathbf{R}^k \to \mathbf{R}^m$ *is a measurable function. Let* $\widehat{\theta}_n$ *be random vector taking their values in the domain of* ϕ *that converge almost surely to* θ. *If* $\sqrt{n}(\widehat{\theta}_n - \theta) \to_d T$, *and* $\sqrt{n}(\widehat{\theta}_n^* - \widehat{\theta}_n) \to_d T$ *conditionally almost surely, then both* $\sqrt{n}(\phi(\widehat{\theta}_n) - \phi(\theta)) \to_d \phi'_\theta(T)$ *and* $\sqrt{n}(\phi(\widehat{\theta}_n^*) - \phi(\widehat{\theta}_n)) \to_d \phi'_\theta(T)$ *conditionally almost surely.*

4. Main Result

We now address our main result. The Yule-Walker equation system for the AR(2) model is

$$\begin{pmatrix} \sum_{t=1}^n X_t^2 & \sum_{t=2}^n X_t X_{t-1} \\ \sum_{t=2}^n X_t X_{t-1} & \sum_{t=1}^n X_t^2 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} \sum_{t=2}^n X_t X_{t-1} \\ \sum_{t=3}^n X_t X_{t-2} \end{pmatrix},$$

or

$$\begin{aligned} \theta_1 \gamma_0 + \theta_2 \gamma_1 &= \gamma_1 \\ \theta_1 \gamma_1 + \theta_2 \gamma_0 &= \gamma_2. \end{aligned}$$

Dividing both sides by $\gamma_0 > 0$ we obtain

$$\begin{aligned} \theta_1 + \theta_2 \rho_1 &= \rho_1 \\ \theta_1 \rho_1 + \theta_2 &= \rho_2. \end{aligned}$$

By the moment method, we obtain the estimator for $\boldsymbol{\theta} = (\theta_1, \theta_2)^T$ as follows:

$$\widehat{\boldsymbol{\theta}} = \begin{pmatrix} \widehat{\theta}_1 \\ \widehat{\theta}_2 \end{pmatrix} = \begin{pmatrix} 1 & \widehat{\rho}_1 \\ \widehat{\rho}_1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \widehat{\rho}_1 \\ \widehat{\rho}_2 \end{pmatrix} = \frac{1}{1 - \widehat{\rho}_1^2} \begin{pmatrix} \widehat{\rho}_1 - \widehat{\rho}_1 \widehat{\rho}_2 \\ -\widehat{\rho}_1^2 + \widehat{\rho}_2 \end{pmatrix}. \tag{5}$$

The estimator $\widehat{\theta}_1$ and $\widehat{\theta}_2$ can be described as follows:

$$\widehat{\theta}_1 = \frac{\widehat{\rho}_1 - \widehat{\rho}_1 \widehat{\rho}_2}{1 - \widehat{\rho}_1^2} = \frac{\sum_{t=2}^n X_t X_{t-1} (\sum_{t=1}^n X_t^2 - \sum_{t=3}^n X_t X_{t-2})}{(\sum_{t=1}^n X_t^2)^2 - (\sum_{t=2}^n X_t X_{t-1})^2}, \tag{6}$$

and

$$\widehat{\theta}_2 = \frac{\widehat{\rho}_1^2 + \widehat{\rho}_2}{1 - \widehat{\rho}_1^2} = \frac{-(\sum_{t=2}^n X_t X_{t-1})^2 + \sum_{t=1}^n X_t^2 \sum_{t=3}^n X_t X_{t-2}}{(\sum_{t=1}^n X_t^2)^2 - (\sum_{t=2}^n X_t X_{t-1})^2}. \tag{7}$$

The estimator $\widehat{\boldsymbol{\theta}} = (\widehat{\theta}_1, \widehat{\theta}_2)^T$ can be expressed as

$$\phi\Big(\sum_{t=1}^n X_t^2, \sum_{t=2}^n X_t X_{t-1}, \sum_{t=3}^n X_t X_{t-2}\Big)$$

for a measurable map $\phi : \mathbf{R}^3 \to \mathbf{R}^2$ defined as

$$\phi(u, v, w) = (\phi_1(u, v, w), \phi_2(u, v, w))^T,$$

where $\phi_1, \phi_2 : \mathbf{R}^3 \to \mathbf{R}$ would be measurable functions defined as

$$\phi_1(u,v,w) = \frac{v(u-w)}{u^2 - v^2} \text{ and } \phi_2(u,v,w) = \frac{-v^2 + uw}{u^2 - v^2}. \tag{8}$$

It is obvious that the functions ϕ_1 and ϕ_2 are differentiable, with the derivative matrix for ϕ_1 is

$$
\begin{aligned}
\phi_1' &= \begin{pmatrix} \frac{\partial}{\partial u}\phi_1(u,v,w) & \frac{\partial}{\partial v}\phi_1(u,v,w) & \frac{\partial}{\partial w}\phi_1(u,v,w) \end{pmatrix} \\
&= \begin{pmatrix} -\frac{v(u^2+v^2-2uw)}{(u^2-v^2)^2} & \frac{(u-w)(u^2+v^2)}{(u^2-v^2)^2} & \frac{-v}{u^2-v^2} \end{pmatrix},
\end{aligned}
$$

and

$$\phi_1'{}_{(\gamma_X(0),\gamma_X(1),\gamma_X(2))} =$$

$$\begin{pmatrix} \frac{-\gamma_X(1)\left(\gamma_X(0)^2+\gamma_X(1)^2-2\gamma_X(0)\gamma_X(2)\right)}{\left(\gamma_X(0)^2-\gamma_X(1)^2\right)^2} & \frac{\left(\gamma_X(0)-\gamma_X(2)\right)\left(\gamma_X(0)^2+\gamma_X(1)^2\right)}{\left(\gamma_X(0)^2-\gamma_X(1)^2\right)^2} & \frac{-\gamma_X(1)}{\gamma_X(0)^2-\gamma_X(1)^2} \end{pmatrix}.$$

While, the derivative matrix for ϕ_2 is

$$
\begin{aligned}
\phi_2' &= \begin{pmatrix} \frac{\partial}{\partial u}\phi_2(u,v,w) & \frac{\partial}{\partial v}\phi_2(u,v,w) & \frac{\partial}{\partial w}\phi_2(u,v,w) \end{pmatrix} \\
&= \begin{pmatrix} \frac{2uv^2-u^2w-v^2w}{(u^2-v^2)^2} & \frac{2uv(w-u)}{(u^2-v^2)^2} & \frac{u}{u^2-v^2} \end{pmatrix},
\end{aligned}
$$

and

$$\phi_2'{}_{(\gamma_X(0),\gamma_X(1),\gamma_X(2))} =$$

$$\begin{pmatrix} \frac{2\gamma_X(0)\gamma_X(1)^2-\gamma_X(0)^2\gamma_X(2)-\gamma_X(1)^2\gamma_X(2)}{\left(\gamma_X(0)^2-\gamma_X(1)^2\right)^2} & \frac{2\gamma_X(0)\gamma_X(1)\left(\gamma_X(2)-\gamma_X(0)\right)}{\left(\gamma_X(0)^2-\gamma_X(1)^2\right)^2} & \frac{\gamma_X(0)}{\gamma_X(0)^2-\gamma_X(1)^2} \end{pmatrix}.$$

The next step, we investigate the asymptotic distribution of the random variable $\widehat{\boldsymbol{\theta}}^* = (\widehat{\theta}_1^*, \widehat{\theta}_2^*)^T$, the bootstrapped version of $\widehat{\boldsymbol{\theta}}$. For simplicity of notation, let

$$
\begin{aligned}
A_1 &= \frac{-\gamma_X(1)(\gamma_X(0)^2 + \gamma_X(1)^2 - 2\gamma_X(0)\gamma_X(2))}{\left(\gamma_X(0)^2 - \gamma_X(1)^2\right)^2}, \\
A_2 &= \frac{(\gamma_X(0) - \gamma_X(2))(\gamma_X(0)^2 + \gamma_X(1)^2)}{\left(\gamma_X(0)^2 - \gamma_X(1)^2\right)^2}, \\
A_3 &= \frac{-\gamma_X(1)}{\gamma_X(0)^2 - \gamma_X(1)^2}, \\
B_1 &= \frac{2\gamma_X(0)\gamma_X(1)^2 - \gamma_X(0)^2\gamma_X(2) - \gamma_X(1)^2\gamma_X(2)}{\left(\gamma_X(0)^2 - \gamma_X(1)^2\right)^2}, \\
B_2 &= \frac{2\gamma_X(0)\gamma_X(1)(\gamma_X(2) - \gamma_X(0))}{\left(\gamma_X(0)^2 - \gamma_X(1)^2\right)^2}, \\
B_3 &= \frac{\gamma_X(0)}{\gamma_X(0)^2 - \gamma_X(1)^2}.
\end{aligned}
$$

By applying Theorem 3.1, we obtain

4

$$\sqrt{n}\left(\phi\left(\frac{1}{n}\sum_{t=1}^{n}X_t^2, \frac{1}{n}\sum_{t=2}^{n}X_{t-1}X_t, \frac{1}{n}\sum_{t=3}^{n}X_{t-2}X_t\right) - \phi(\gamma_X(0), \gamma_X(1), \gamma_X(2))\right)$$

$$\left(\begin{array}{ccc} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{array}\right)\left(\begin{array}{c} \sqrt{n}\left(\frac{1}{n}\sum_{t=1}^{n}X_t^2 - \gamma_X(0)\right) \\ \sqrt{n}\left(\frac{1}{n}\sum_{t=2}^{n}X_{t-1}X_t - \gamma_X(1)\right) \\ \sqrt{n}\left(\frac{1}{n}\sum_{t=3}^{n}X_{t-2}X_t - \gamma_X(2)\right) \end{array}\right) + \mathbf{o}_p(1). \tag{9}$$

According to Theorem 2.2, the multivariate limiting distribution of the random vector $\left(\frac{1}{n}\sum_{t=1}^{n}X_t^2, \frac{1}{n}\sum_{t=2}^{n}X_tX_{t-1}, \frac{1}{n}\sum_{t=3}^{n}X_tX_{t-2}\right)^T$ is

$$\sqrt{n}\left(\left(\begin{array}{c} \frac{1}{n}\sum_{t=1}^{n}X_t^2 \\ \frac{1}{n}\sum_{t=2}^{n}X_tX_{t-1} \\ \frac{1}{n}\sum_{t=3}^{n}X_tX_{t-2} \end{array}\right) - \left(\begin{array}{c} \gamma_X(0) \\ \gamma_X(1) \\ \gamma_X(2) \end{array}\right)\right)$$

$$\to_d N_3\left(\left(\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right), \left(\begin{array}{ccc} V_{0,0} & V_{0,1} & V_{0,2} \\ V_{1,0} & V_{1,1} & V_{1,2} \\ V_{2,0} & V_{2,1} & V_{2,2} \end{array}\right)\right). \tag{10}$$

In view of Theorem 3.1, if $(Z_1, Z_2, Z_3)^T$ possesses the multivariate normal distribution as in (10), then

$$\left(\begin{array}{ccc} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{array}\right)\left(\begin{array}{c} \sqrt{n}\left(\frac{1}{n}\sum_{t=1}^{n}X_t^2 - \gamma_X(0)\right) \\ \sqrt{n}\left(\frac{1}{n}\sum_{t=2}^{n}X_{t-1}X_t - \gamma_X(1)\right) \\ \sqrt{n}\left(\frac{1}{n}\sum_{t=3}^{n}X_{t-2}X_t - \gamma_X(2)\right) \end{array}\right)$$

$$\to_d \left(\begin{array}{ccc} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{array}\right)\left(\begin{array}{c} Z_1 \\ Z_2 \\ Z_3 \end{array}\right) \sim N_2\left(\left(\begin{array}{c} 0 \\ 0 \end{array}\right), \left(\begin{array}{cc} \tau_1^2 & \tau_{12} \\ \tau_{21} & \tau_2^2 \end{array}\right)\right).$$

Hence, by Theorem 3.1 we deduce that

$$\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}) =$$

$$\sqrt{n}\left(\begin{array}{c} \phi_1\left(\frac{1}{n}\sum_{t=1}^{n}X_t^2, \frac{1}{n}\sum_{t=2}^{n}X_{t-1}X_t, \frac{1}{n}\sum_{t=3}^{n}X_{t-2}X_t\right) - \phi_1(\gamma_X(0), \gamma_X(1), \gamma_X(2)) \\ \phi_2\left(\frac{1}{n}\sum_{t=1}^{n}X_t^2, \frac{1}{n}\sum_{t=2}^{n}X_{t-1}X_t, \frac{1}{n}\sum_{t=3}^{n}X_{t-2}X_t\right) - \phi_2(\gamma_X(0), \gamma_X(1), \gamma_X(2)) \end{array}\right)$$

$$\to_d N_2\left(\left(\begin{array}{c} 0 \\ 0 \end{array}\right), \left(\begin{array}{cc} \tau_1^2 & \tau_{12} \\ \tau_{21} & \tau_2^2 \end{array}\right)\right),$$

where

$$\begin{aligned} \tau_1^2 &= Var(A_1Z_1 + A_2Z_2 + A_3Z_3) \\ &= A_1^2V_{0,0} + A_2^2V_{1,1} + A_3^2V_{2,2} + 2A_1A_2V_{0,1} + 2A_1A_3V_{0,2} + 2A_2A_3V_{1,2}, \\ \tau_2^2 &= Var(B_1Z_1 + B_2Z_2 + B_3Z_3) \\ &= B_1^2V_{0,0} + B_2^2V_{1,1} + B_3^2V_{2,2} + 2B_1B_2V_{0,1} + 2B_1B_3V_{0,2} + 2B_2B_3V_{1,2}, \\ \tau_{12} = \tau_{21} &= Cov(A_1Z_1 + A_2Z_2 + A_3Z_3, B_1Z_1 + B_2Z_2 + B_3Z_3). \end{aligned}$$

An analogous representation holds for the bootstrapped version (see, *e.g* [3], [9]). The residuals bootstrapping procedure used was proposed in [7] as follows. Define the residuals

$\widehat{\varepsilon}_t = X_t - \left(\widehat{\theta}_1 X_{t-1} + \widehat{\theta}_2 X_{t-2}\right), \quad t = 3, 4, \ldots, n$. The bootstrap sample $X_1^*, X_2^*, \ldots, X_n^*$ are obtained by resampling without replacement from the residuals $\widehat{\varepsilon}_3, \widehat{\varepsilon}_4, \ldots, \widehat{\varepsilon}_n$. Let $\widehat{F}_n$ be the empirical distribution of $\widehat{\varepsilon}_3, \widehat{\varepsilon}_4, \ldots, \widehat{\varepsilon}_n$, puts mass $1/n$ at each of the computed residuals. Now, the sequence of bootstrap residuals $\varepsilon_3^*, \varepsilon_4^*, \ldots, \varepsilon_n^*$ be conditionally independent with common distribution $\widehat{F}_n$. If given that $X_j^* = X_j, \quad j = 1, 2$, as initial bootstrapping sample then we obtain $X_t^* = \widehat{\theta}_1 X_{t-1} + \widehat{\theta}_2 X_{t-2} + \varepsilon_t^*, t = 3, 4, \ldots, n$. Both [3] and [7] proved that the residuals bootstrapping work well when it is applied to the autoregressive model.

We can see that the estimator $\widehat{\boldsymbol{\theta}}^* = (\widehat{\theta}_1^*, \widehat{\theta}_2^*)^T$ can be written as

$$\phi\Big(\sum_{t=1}^n X_t^{2*}, \sum_{t=2}^n X_t^* X_{t-1}^*, \sum_{t=3}^n X_t^* X_{t-2}^*\Big)$$

for a measurable map $\phi : \mathbf{R}^3 \to \mathbf{R}^2$,

$$\phi(u, v, w) = (\phi_1(u, v, w), \phi_2(u, v, w))^T,$$

with $\phi_1, \phi_2 : \mathbf{R}^3 \to \mathbf{R}$ be measurable functions as defined in (8). The function ϕ_1 is differentiable with derivative matrix

$\phi'_{1(\widehat{\gamma}_X(0), \widehat{\gamma}_X(1), \widehat{\gamma}_X(2))} =$

$$\left(\frac{-\widehat{\gamma}_X(1)\left(\widehat{\gamma}_X(0)^2 + \widehat{\gamma}_X(1)^2 - 2\widehat{\gamma}_X(0)\widehat{\gamma}_X(2)\right)}{\left(\widehat{\gamma}_X(0)^2 - \widehat{\gamma}_X(1)^2\right)^2} \quad \frac{\left(\widehat{\gamma}_X(0) - \widehat{\gamma}_X(2)\right)\left(\widehat{\gamma}_X(0)^2 + \widehat{\gamma}_X(1)^2\right)}{\left(\widehat{\gamma}_X(0)^2 - \widehat{\gamma}_X(1)^2\right)^2} \quad \frac{-\widehat{\gamma}_X(1)}{\widehat{\gamma}_X(0)^2 - \widehat{\gamma}_X(1)^2} \right).$$

Also, the function ϕ_2 is differentiable with derivative matrix

$\phi'_{2(\widehat{\gamma}_X(0), \widehat{\gamma}_X(1), \widehat{\gamma}_X(2))} =$

$$\left(\frac{2\widehat{\gamma}_X(0)\widehat{\gamma}_X(1)^2 - \widehat{\gamma}_X(0)^2\widehat{\gamma}_X(2) - \widehat{\gamma}_X(1)^2\widehat{\gamma}_X(2)}{\left(\widehat{\gamma}_X(0)^2 - \widehat{\gamma}_X(1)^2\right)^2} \quad \frac{2\widehat{\gamma}_X(0)\widehat{\gamma}_X(1)\left(\widehat{\gamma}_X(2) - \widehat{\gamma}_X(0)\right)}{\left(\widehat{\gamma}_X(0)^2 - \widehat{\gamma}_X(1)^2\right)^2} \quad \frac{\widehat{\gamma}_X(0)}{\widehat{\gamma}_X(0)^2 - \widehat{\gamma}_X(1)^2} \right).$$

Now, we are ready to investigate the asymptotic distribution of the random vector $\widehat{\boldsymbol{\theta}}^* = (\widehat{\theta}_1^*, \widehat{\theta}_2^*)^T$, bootstrap version of $\widehat{\boldsymbol{\theta}} = (\widehat{\theta}_1, \widehat{\theta}_2)^T$. For shake the simplicity, let

$$
\begin{aligned}
C_1 &= \frac{-\widehat{\gamma}_X(1)(\widehat{\gamma}_X(0)^2 + \widehat{\gamma}_X(1)^2 - 2\widehat{\gamma}_X(0)\widehat{\gamma}_X(2))}{(\widehat{\gamma}_X(0)^2 - \widehat{\gamma}_X(1)^2)^2}, \\
C_2 &= \frac{(\widehat{\gamma}_X(0) - \widehat{\gamma}_X(2))(\widehat{\gamma}_X(0)^2 + \widehat{\gamma}_X(1)^2)}{(\widehat{\gamma}_X(0)^2 - \widehat{\gamma}_X(1)^2)^2}, \\
C_3 &= \frac{-\widehat{\gamma}_X(1)}{\widehat{\gamma}_X(0)^2 - \widehat{\gamma}_X(1)^2}, \\
D_1 &= \frac{2\widehat{\gamma}_X(0)\widehat{\gamma}_X(1)^2 - \widehat{\gamma}_X(0)^2\widehat{\gamma}_X(2) - \widehat{\gamma}_X(1)^2\widehat{\gamma}_X(2)}{(\widehat{\gamma}_X(0)^2 - \widehat{\gamma}_X(1)^2)^2}, \\
D_2 &= \frac{2\widehat{\gamma}_X(0)\widehat{\gamma}_X(1)(\widehat{\gamma}_X(2) - \widehat{\gamma}_X(0))}{(\widehat{\gamma}_X(0)^2 - \widehat{\gamma}_X(1)^2)^2}, \\
D_3 &= \frac{\widehat{\gamma}_X(0)}{\widehat{\gamma}_X(0)^2 - \widehat{\gamma}_X(1)^2}.
\end{aligned}
$$

By applying Theorem 3.1, we obtain

$$\sqrt{n}\left(\phi\left(\tfrac{1}{n}\sum_{t=1}^{n}X_t^{2*},\tfrac{1}{n}\sum_{t=2}^{n}X_{t-1}^*X_t^*,\tfrac{1}{n}\sum_{t=3}^{n}X_{t-2}^*X_t^*\right)-\phi\left(\widehat{\gamma}_X(0),\widehat{\gamma}_X(1),\widehat{\gamma}_X(2)\right)\right)$$

$$=\left(\begin{array}{c}\phi'_{1(\widehat{\gamma}_X(0),\widehat{\gamma}_X(1),\widehat{\gamma}_X(2))}\\ \phi'_{2(\widehat{\gamma}_X(0),\widehat{\gamma}_X(1),\widehat{\gamma}_X(2))}\end{array}\right)\left(\begin{array}{c}\sqrt{n}\left(\tfrac{1}{n}\sum_{t=1}^{n}X_t^{2*}-\widehat{\gamma}_X(0)\right)\\ \sqrt{n}\left(\tfrac{1}{n}\sum_{t=2}^{n}X_{t-1}^*X_t^*-\widehat{\gamma}_X(1)\right)\\ \sqrt{n}\left(\tfrac{1}{n}\sum_{t=3}^{n}X_{t-2}^*X_t^*-\widehat{\gamma}_X(2)\right)\end{array}\right)+\mathbf{o}_p(1)$$

$$=\left(\begin{array}{ccc}C_1 & C_2 & C_3\\ D_1 & D_2 & D_3\end{array}\right)\left(\begin{array}{c}\sqrt{n}\left(\tfrac{1}{n}\sum_{t=1}^{n}X_t^{2*}-\widehat{\gamma}_X(0)\right)\\ \sqrt{n}\left(\tfrac{1}{n}\sum_{t=2}^{n}X_{t-1}^*X_t^*-\widehat{\gamma}_X(1)\right)\\ \sqrt{n}\left(\tfrac{1}{n}\sum_{t=3}^{n}X_{t-2}^*X_t^*-\widehat{\gamma}_X(2)\right)\end{array}\right)+\mathbf{o}_p(1).$$

According to Theorem 2.2, the multivariate limiting distribution of random variables $\left(\tfrac{1}{n}\sum_{t=1}^{n}X_t^{2*},\tfrac{1}{n}\sum_{t=2}^{n}X_t^*X_{t-1}^*,\tfrac{1}{n}\sum_{t=3}^{n}X_t^*X_{t-2}^*\right)^T$ is

$$\sqrt{n}\left(\left(\begin{array}{c}\tfrac{1}{n}\sum_{t=1}^{n}X_t^{2*}\\ \tfrac{1}{n}\sum_{t=2}^{n}X_t^*X_{t-1}^*\\ \tfrac{1}{n}\sum_{t=3}^{n}X_t^*X_{t-2}^*\end{array}\right)-\left(\begin{array}{c}\widehat{\gamma}_X(0)\\ \widehat{\gamma}_X(1)\\ \widehat{\gamma}_X(2)\end{array}\right)\right)$$

$$\to_d N_3\left(\left(\begin{array}{c}0\\0\\0\end{array}\right),\left(\begin{array}{ccc}V_{0,0}^* & V_{0,1}^* & V_{0,2}^*\\ V_{1,0}^* & V_{1,1}^* & V_{1,2}^*\\ V_{2,0}^* & V_{2,1}^* & V_{2,2}^*\end{array}\right)\right). \tag{11}$$

Meanwhile, by Theorem 3.1, if $(W_1,W_2,W_3)^T$ posses multivariate normal distribution as in (11), then

$$\left(\begin{array}{ccc}C_1 & C_2 & C_3\\ D_1 & D_2 & D_3\end{array}\right)\left(\begin{array}{c}\sqrt{n}\left(\tfrac{1}{n}\sum_{t=1}^{n}X_t^{2*}-\widehat{\gamma}_X(0)\right)\\ \sqrt{n}\left(\tfrac{1}{n}\sum_{t=2}^{n}X_{t-1}^*X_t^*-\widehat{\gamma}_X(1)\right)\\ \sqrt{n}\left(\tfrac{1}{n}\sum_{t=3}^{n}X_{t-2}^*X_t^*-\widehat{\gamma}_X(2)\right)\end{array}\right)$$

$$\to_d\left(\begin{array}{ccc}C_1 & C_2 & C_3\\ D_1 & D_2 & D_3\end{array}\right)\left(\begin{array}{c}W_1\\ W_2\\ W_3\end{array}\right)\sim N_2\left(\left(\begin{array}{c}0\\0\end{array}\right),\left(\begin{array}{cc}\tau_1^{2*} & \tau_{12}^*\\ \tau_{21}^* & \tau_2^{2*}\end{array}\right)\right),$$

where $\tau_1^{2*},\tau_2^{2*},\tau_{12}^*$ dan τ_{21}^* are bootstrap version of $\tau_1^2,\tau_2^2,\tau_{12}$ dan τ_{21} respectively. Hence, by Theorem 3.2, we conclude that

$$\sqrt{n}(\widehat{\boldsymbol{\theta}}^*-\widehat{\boldsymbol{\theta}})\to_d N_2\left(\left(\begin{array}{c}0\\0\end{array}\right),\left(\begin{array}{cc}\tau_1^{2*} & \tau_{12}^*\\ \tau_{21}^* & \tau_2^{2*}\end{array}\right)\right).$$

5. Conclusions
We conclude that the bootstrap parameter estimator of the second order autoregressive model is asymptotic, and has limiting distribution to the bivariate normal distribution.

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