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Asymptotic Distribution of the Bootstrap Parameter Estimator of an AR(2) Model

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Abstract. This paper is the extension of our research about asymptotic distribution of the bootstrap parameter estimator for the AR(1) model. We investigate the asymptotic distribution of the bootstrap parameter estimator of a second order autoregressive AR(2) model by applying the delta method. The asymptotic distribution is the crucial property in inference of statistics. We conclude that the bootstrap parameter estimator of the AR(2) model asymptotically converges in distribution to the bivariate normal distribution.

keywords

1. Introduction

Consider the following stationary second order autoregressive AR(2) process:

$$X_t = \theta_1 X_{t-1} + \theta_2 X_{t-2} + \epsilon_t, \tag{1}$$

where ϵ_t is a zero mean white noise process with constant variance σ^2. Let the vector $\widehat{\boldsymbol{\theta}} = (\widehat{\theta_1}, \widehat{\theta_2})^T$ is the estimator of the parameter vector $\boldsymbol{\theta} = (\theta_1, \theta_2)^T$ of (1) and $\widehat{\boldsymbol{\theta}}^{\boldsymbol{*}}$ be the bootstrap version of $\widehat{\boldsymbol{\theta}}$. Studying of estimation of the unknown parameter involves: (i) what estimator should be used? (ii) having chosen a particular estimator, is this consistent? (iii) how accurate is the chosen estimator? (iv) what is the asymptotic behaviour of such estimator? (v) what is the method used in proving the asymptotic properties?

Bootstrap is a general methodology for answering the second and third questions, while the delta method is one of tools used to answer the last two questions. Consistency theory is needed to ensure that the estimator is consistent to the actual parameter as desired, and thereof the asymptotic behaviour of such estimator will be studied. The consistency theories of parameter of autoregressive model have studied in [1, 4, 5], and for bootstrap version of the same topic, [1,4-5] see *e.g.* [2, 6, 7, 8, 9, 11]. They deal with the bootstrap approximation in various senses [2, 6-9,11] (*e.g.,* consistency of estimator, simulation results, limiting distribution, applying of Edgeworth expansions, etc.), and they reported that the bootstrap works usually very well. The accuracy of the bootstrapping method for autoregressive model studied in [3, 10]. They showed that the parameter estimates of the autoregressive model can be bootstrapped with accuracy that outperforms the normal approximation. The asymptotic result for the AR(1) model has been exhibited in [12]. We concluded that the bootstrap parameter estimator for the AR(1) model converges in distribution to the normal distribution. A good perform of the bootstrap estimator is applied to study the asymptotic distribution of $\widehat{\boldsymbol{\theta}}^{\boldsymbol{*}}$ using the delta method. We describe the asymptotic distribution of the autocovariance function and investigate the bootstrap limiting distribution of $\widehat{\boldsymbol{\theta}}^*$. Section 2 reviews the asymptotic distribution of estimator of mean and autocovariance function for the autoregression model. Section 3 describes the bootstrap and delta method. Section 4 deals with the main result, *i.e.* the asymptotic distribution of $\widehat{\boldsymbol{\theta}}^*$ by applying the delta method. Section 5 briefly describes the conclusions of the paper.

2. Estimator of Mean and Autocovariance for the Autoregressive Model

Suppose we have the observed values $X_1, X_2, \ldots, X_n$ from the stationary AR(2) process. A natural estimators for parameters mean, covariance and correlation function are $\widehat{\mu}_n = \overline{X}_n = \frac{1}{n}\sum_{t=1}^{n} X_t$, $\widehat{\gamma}_n(h) = \frac{1}{n}\sum_{t=1}^{n-h}(X_{t+h}-\overline{X}_n)(X_t-\overline{X}_n)$, and $\widehat{\rho}_n(h) = \widehat{\gamma}_n/\widehat{\gamma}_n(0)$ respectively. These all three estimators are consistent (see, *e.g.*, [4, 14]). The following theorem describes the property of the estimator $\overline{X}_n$, is stated in [4].

Theorem 2.1 *If $\{X_t\}$ is stationary process with mean μ and autocovariance function $\gamma(\cdot)$, then as $n \to \infty$,*

$$Var(\overline{X}_n) = E(\overline{X}_n - \mu)^2 \to 0 \quad \textit{if } \gamma(n) \to 0,$$

and

$$nE(\overline{X}_n - \mu)^2 \to \sum_{j=-\infty}^{\infty} \gamma(h) \quad \textit{if } \sum_{j=-\infty}^{\infty} |\gamma(h)| < \infty.$$

It is not a loss of generality to assume that $\mu_X = 0$. Under some conditions (see, *e.g.,* [14]), the sample autocovariance function can be written as

$$\widehat{\gamma}_n(h) = \frac{1}{n}\sum_{t=1}^{n-h} X_{t+h}X_t + O_p(1/n). \tag{2}$$

The asymptotic behaviour of the sequence $\sqrt{n}\,(\widehat{\gamma}_n(h) - \gamma_X(h))$ depends only on $n^{-1}\sum_{t=1}^{n-h} X_{t+h}X_t$. Note that a change of $n-h$ by n or vice versa, is asymptotically negligible, so that, for simplicity of notation, to study the behavior of (2) we can equivalently study the average

$$\widetilde{\gamma}_n(h) = \frac{1}{n}\sum_{t=1}^{n} X_{t+h}X_t. \tag{3}$$

Both (2) and (3) are unbiased estimators of $E(X_{t+h}X_t) = \gamma_X(h)$, under the condition that $\mu_X = 0$. Their asymptotic distribution then can be derived by applying a central limit theorem to the averages $\overline{Y}_n$ of the variables $Y_t = X_{t+h}X_t$. As in [14], the autocovariance function of the series Y_t can be written as

$$V_{h,h} = \kappa_4(\varepsilon)\gamma_X(h)^2 + \sum_g \gamma_X(g)^2 + \sum_g \gamma_X(g+h)\gamma_X(g-h), \tag{4}$$

where $\kappa_4(\varepsilon) = E(\varepsilon_1^4) - 3(E(\varepsilon_1^2))^2$, the fourth cumulant of ε_t. The following theorem is due to [14] that gives the asymptotic distribution of the sequence $\sqrt{n}\,(\widehat{\gamma}_n(h) - \gamma_X(h))$.

Theorem 2.2 *If $X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j}$ holds for an i.i.d. sequence ε_t with mean zero and $E(\varepsilon_t^4) < \infty$ and numbers ψ_j with $\sum_j |\psi_j| < \infty$, then*

$$\sqrt{n}\,(\widehat{\gamma}_n(h) - \gamma_X(h)) \to_d N(0, V_{h,h}).$$

3. Bootstrap and Delta Method

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a population with common distribution F, and let $T(X_1, X_2, \ldots, X_n; F)$ be the specified random variable or statistic of interest, possibly depending upon the unknown distribution F. Let F_n denote the empirical distribution function of the random sample $X_1, X_2, \ldots, X_n$, *i.e.,* the distribution putting probability 1/n at each of the points $X_1, X_2, \ldots, X_n$. A bootstrap sample is defined to be a random sample of size n drawn from F_n, say $X^* = X_1^*, X_2^*, \ldots, X_n^*$. The bootstrap method is to approximate the distribution of $T(X_1, X_2, \ldots, X_n; F)$ under F by that of $T(X_1^*, X_2^*, \ldots, X_n^*; F_n)$ under F_n. n should in italics

Let a functional T is defined as $T(X_1, X_2, \ldots, X_n; F) = \sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})$, where $\widehat{\boldsymbol{\theta}}$ is the estimator for the coefficient $\boldsymbol{\theta}$ of a stationary AR(2) model. The bootstrap version of T is $T(X_1^*, X_2^*, \ldots, X_n^*; F_n) = \sqrt{n}(\widehat{\boldsymbol{\theta}}^* - \widehat{\boldsymbol{\theta}})$, where $\widehat{\boldsymbol{\theta}}^*$ is a bootstrap version of $\widehat{\boldsymbol{\theta}}$ computed from sample bootstrap $X_1^*, X_2^*, \ldots, X_n^*$. The residuals bootstrapping procedure for the time series data to obtain $X_1^*, X_2^*, \ldots, X_n^*$ was proposed in [7]. When we want to investigate the asymptotic distribution of bootstrap estimator $\widehat{\boldsymbol{\theta}}^*$, we investigate the distribution of $\sqrt{n}(\widehat{\boldsymbol{\theta}}^* - \widehat{\boldsymbol{\theta}})$ contrast to the distribution of $\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})$. Thus, the bootstrap is a device for estimating $P_F(\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \le x)$ by $P_{F_n}(\sqrt{n}(\widehat{\boldsymbol{\theta}}^* - \widehat{\boldsymbol{\theta}}) \le x)$. We propose the delta method in estimating for such distribution.

The delta method is useful to deduce the limit law of $\phi(T_n) - \phi(\theta)$ from that of $T_n - \theta$, which is guaranteed by the following theorem, as stated in [13].

Theorem 3.1 *Let $\phi : \mathbf{D}_\phi \subset \mathbf{R}^k \to \mathbf{R}^m$ be a map defined on a subset of $\mathbf{R}^k$ and differentiable at θ. Let T_n be random vector taking their values in the domain of ϕ. If $r_n(T_n - \theta) \to_d T$ for numbers $r_n \to \infty$, then $r_n(\phi(T_n) - \phi(\theta)) \to_d \phi_\theta'(T)$. Moreover, $\left| r_n(\phi(T_n) - \phi(\theta)) - \phi_\theta'(r_n(T_n - \theta)) \right| \to_p 0$.*

Assume that $\widehat{\theta}_n$ is a statistic, and that ϕ is a given differentiable map. The bootstrap estimator for the distribution of $\phi(\widehat{\theta}_n - \phi(\theta)$ is $\phi(\widehat{\theta}_n^* - \phi(\widehat{\theta}_n)$. If the bootstrap is consistent for estimating the distribution of $\sqrt{n}(\widehat{\theta}_n - \theta)$, then it is also consistent for estimating the distribution of $\sqrt{n}(\phi(\widehat{\theta}_n) - \phi(\theta))$, as given in the following theorem. The theorem is due to [13].

Theorem 3.2 (Delta Method For Bootstrap) *Let $\phi : \mathbf{R}^k \to \mathbf{R}^m$ be a measurable map defined and continuously differentiable in a neighborhood of θ. Let $\widehat{\theta}_n$ be random vector taking their values in the domain of ϕ that converge almost surely to θ. If $\sqrt{n}(\widehat{\theta}_n - \theta) \to_d T$, and $\sqrt{n}(\widehat{\theta}_n^* - \widehat{\theta}_n) \to_d T$ conditionally almost surely, then both $\sqrt{n}(\phi(\widehat{\theta}_n) - \phi(\theta)) \to_d \phi_\theta'(T)$ and $\sqrt{n}(\phi(\widehat{\theta}_n^*) - \phi(\widehat{\theta}_n)) \to_d \phi_\theta'(T)$ conditionally almost surely.*

4. Main Result

We now address our main result. The Yule-Walker equation system for the AR(2) model is

$$\begin{pmatrix} \sum_{t=1}^n X_t^2 & \sum_{t=2}^n X_t X_{t-1} \\ \sum_{t=2}^n X_t X_{t-1} & \sum_{t=1}^n X_t^2 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} \sum_{t=2}^n X_t X_{t-1} \\ \sum_{t=3}^n X_t X_{t-2} \end{pmatrix},$$

or

$$\begin{aligned} \theta_1\gamma_0 + \theta_2\gamma_1 &= \gamma_1 \\ \theta_1\gamma_1 + \theta_2\gamma_0 &= \gamma_2. \end{aligned}$$

Dividing both sides by $\gamma_0 > 0$ we obtain

$$\begin{aligned} \theta_1 + \theta_2\rho_1 &= \rho_1 \\ \theta_1\rho_1 + \theta_2 &= \rho_2. \end{aligned}$$

By the moment method, we obtain the estimator for $\boldsymbol{\theta} = (\theta_1, \theta_2)^T$ as follows:

$$\widehat{\boldsymbol{\theta}} = \begin{pmatrix} \widehat{\theta}_1 \\ \widehat{\theta}_2 \end{pmatrix} = \begin{pmatrix} 1 & \widehat{\rho}_1 \\ \widehat{\rho}_1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \widehat{\rho}_1 \\ \widehat{\rho}_2 \end{pmatrix} = \frac{1}{1-\widehat{\rho}_1^2} \begin{pmatrix} \widehat{\rho}_1 - \widehat{\rho}_1\widehat{\rho}_2 \\ -\widehat{\rho}_1^2 + \widehat{\rho}_2 \end{pmatrix}. \tag{5}$$

The estimator $\widehat{\theta}_1$ and $\widehat{\theta}_2$ can be described as follows:

$$\widehat{\theta}_1 = \frac{\widehat{\rho}_1 - \widehat{\rho}_1\widehat{\rho}_2}{1-\widehat{\rho}_1^2} = \frac{\sum_{t=2}^n X_t X_{t-1}(\sum_{t=1}^n X_t^2 - \sum_{t=3}^n X_t X_{t-2})}{(\sum_{t=1}^n X_t^2)^2 - (\sum_{t=2}^n X_t X_{t-1})^2}, \tag{6}$$

and

$$\widehat{\theta}_2 = \frac{\widehat{\rho}_1^2 + \widehat{\rho}_2}{1-\widehat{\rho}_1^2} = \frac{-(\sum_{t=2}^n X_t X_{t-1})^2 + \sum_{t=1}^n X_t^2 \sum_{t=3}^n X_t X_{t-2}}{(\sum_{t=1}^n X_t^2)^2 - (\sum_{t=2}^n X_t X_{t-1})^2}. \tag{7}$$

The estimator $\widehat{\boldsymbol{\theta}} = (\widehat{\theta}_1, \widehat{\theta}_2)^T$ can be expressed as

$$\phi\Big(\sum_{t=1}^n X_t^2, \sum_{t=2}^n X_t X_{t-1}, \sum_{t=3}^n X_t X_{t-2}\Big)$$

for a measurable map $\phi : \mathbf{R}^3 \to \mathbf{R}^2$ defined as

$$\phi(u,v,w) = (\phi_1(u,v,w), \phi_2(u,v,w))^T,$$

where $\phi_1, \phi_2 : \mathbf{R}^3 \to \mathbf{R}$ would be measurable functions defined as

$$\phi_1(u,v,w) = \frac{v(u-w)}{u^2-v^2} \text{ and } \phi_2(u,v,w) = \frac{-v^2+uw}{u^2-v^2}. \tag{8}$$

It is obvious that the functions ϕ_1 and ϕ_2 are differentiable, with the derivative matrix for ϕ_1 is

$$\begin{aligned} \phi_1' &= \begin{pmatrix} \frac{\partial}{\partial u}\phi_1(u,v,w) & \frac{\partial}{\partial v}\phi_1(u,v,w) & \frac{\partial}{\partial w}\phi_1(u,v,w) \end{pmatrix} \\ &= \begin{pmatrix} -\frac{v(u^2+v^2-2uw)}{(u^2-v^2)^2} & \frac{(u-w)(u^2+v^2}{(u^2-v^2)^2} & \frac{-v}{u^2-v^2} \end{pmatrix}, \end{aligned}$$

and
$\phi'_{1(\gamma_X(0),\gamma_X(1),\gamma_X(2))} =$

$$\begin{pmatrix} \frac{-\gamma_X(1)\left(\gamma_X(0)^2+\gamma_X(1)^2-2\gamma_X(0)\gamma_X(2)\right)}{\left(\gamma_X(0)^2-\gamma_X(1)^2\right)^2} & \frac{\left(\gamma_X(0)-\gamma_X(2)\right)\left(\gamma_X(0)^2+\gamma_X(1)^2\right)}{\left(\gamma_X(0)^2-\gamma_X(1)^2\right)^2} & \frac{-\gamma_X(1)}{\gamma_X(0)^2-\gamma_X(1)^2} \end{pmatrix}.$$

While, the derivative matrix for ϕ_2 is

$$\begin{aligned} \phi_2' &= \begin{pmatrix} \frac{\partial}{\partial u}\phi_2(u,v,w) & \frac{\partial}{\partial v}\phi_2(u,v,w) & \frac{\partial}{\partial w}\phi_2(u,v,w) \end{pmatrix} \\ &= \begin{pmatrix} \frac{2uv^2-u^2w-v^2w}{(u^2-v^2)^2} & \frac{2uv(w-u)}{(u^2-v^2)^2} & \frac{u}{u^2-v^2} \end{pmatrix}, \end{aligned}$$

and
$\phi'_{2(\gamma_X(0),\gamma_X(1),\gamma_X(2))} =$

$$\begin{pmatrix} \frac{2\gamma_X(0)\gamma_X(1)^2-\gamma_X(0)^2\gamma_X(2)-\gamma_X(1)^2\gamma_X(2)}{\left(\gamma_X(0)^2-\gamma_X(1)^2\right)^2} & \frac{2\gamma_X(0)\gamma_X(1)\left(\gamma_X(2)-\gamma_X(0)\right)}{\left(\gamma_X(0)^2-\gamma_X(1)^2\right)^2} & \frac{\gamma_X(0)}{\gamma_X(0)^2-\gamma_X(1)^2} \end{pmatrix}.$$

The next step, we investigate the asymptotic distribution of the random variable $\widehat{\boldsymbol{\theta}}^* = (\widehat{\theta}_1^*, \widehat{\theta}_2^*)^T$, the bootstrapped version of $\widehat{\boldsymbol{\theta}}$. For simplicity of notation, let

$$\begin{aligned}
A_1 &= \frac{-\gamma_X(1)(\gamma_X(0)^2+\gamma_X(1)^2-2\gamma_X(0)\gamma_X(2))}{(\gamma_X(0)^2-\gamma_X(1)^2)^2}, \\
A_2 &= \frac{(\gamma_X(0)-\gamma_X(2))(\gamma_X(0)^2+\gamma_X(1)^2)}{(\gamma_X(0)^2-\gamma_X(1)^2)^2}, \\
A_3 &= \frac{-\gamma_X(1)}{\gamma_X(0)^2-\gamma_X(1)^2}, \\
B_1 &= \frac{2\gamma_X(0)\gamma_X(1)^2-\gamma_X(0)^2\gamma_X(2)-\gamma_X(1)^2\gamma_X(2)}{(\gamma_X(0)^2-\gamma_X(1)^2)^2}, \\
B_2 &= \frac{2\gamma_X(0)\gamma_X(1)(\gamma_X(2)-\gamma_X(0))}{(\gamma_X(0)^2-\gamma_X(1)^2)^2}, \\
B_3 &= \frac{\gamma_X(0)}{\gamma_X(0)^2-\gamma_X(1)^2}.
\end{aligned}$$

By applying Theorem 3.1, we obtain

$$\sqrt{n}\left(\boldsymbol{\phi}\left(\tfrac{1}{n}\textstyle\sum_{t=1}^n X_t^2, \tfrac{1}{n}\sum_{t=2}^n X_{t-1}X_t, \tfrac{1}{n}\sum_{t=3}^n X_{t-2}X_t\right) - \boldsymbol{\phi}(\gamma_X(0),\gamma_X(1),\gamma_X(2))\right)$$

$$\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{pmatrix} \begin{pmatrix} \sqrt{n}\left(\frac{1}{n}\sum_{t=1}^n X_t^2 - \gamma_X(0)\right) \\ \sqrt{n}\left(\frac{1}{n}\sum_{t=2}^n X_{t-1}X_t - \gamma_X(1)\right) \\ \sqrt{n}\left(\frac{1}{n}\sum_{t=3}^n X_{t-2}X_t - \gamma_X(2)\right) \end{pmatrix} + \mathbf{o}_p(1). \tag{9}$$

According to Theorem 2.2, the multivariate limiting distribution of the random vector $\left(\frac{1}{n}\sum_{t=1}^n X_t^2, \frac{1}{n}\sum_{t=2}^n X_tX_{t-1}, \frac{1}{n}\sum_{t=3}^n X_tX_{t-2}\right)^T$ is

$$\sqrt{n}\left(\begin{pmatrix} \frac{1}{n}\sum_{t=1}^n X_t^2 \\ \frac{1}{n}\sum_{t=2}^n X_tX_{t-1} \\ \frac{1}{n}\sum_{t=3}^n X_tX_{t-2} \end{pmatrix} - \begin{pmatrix} \gamma_X(0) \\ \gamma_X(1) \\ \gamma_X(2) \end{pmatrix}\right)$$

$$\to_d N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} V_{0,0} & V_{0,1} & V_{0,2} \\ V_{1,0} & V_{1,1} & V_{1,2} \\ V_{2,0} & V_{2,1} & V_{2,2} \end{pmatrix}\right). \tag{10}$$

In view of Theorem 3.1, if $(Z_1, Z_2, Z_3)^T$ possesses the multivariate normal distribution as in (10), then

$$\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{pmatrix} \begin{pmatrix} \sqrt{n}\left(\frac{1}{n}\sum_{t=1}^n X_t^2 - \gamma_X(0)\right) \\ \sqrt{n}\left(\frac{1}{n}\sum_{t=2}^n X_{t-1}X_t - \gamma_X(1)\right) \\ \sqrt{n}\left(\frac{1}{n}\sum_{t=3}^n X_{t-2}X_t - \gamma_X(2)\right) \end{pmatrix}$$

$$\to_d \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix} \sim N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \tau_1^2 & \tau_{12} \\ \tau_{21} & \tau_2^2 \end{pmatrix}\right).$$

Hence, by Theorem 3.1 we deduce that

$$\sqrt{n}(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}) =$$

$$\sqrt{n}\begin{pmatrix} \phi_1\Big(\frac{1}{n}\sum_{t=1}^{n}X_t^2, \frac{1}{n}\sum_{t=2}^{n}X_{t-1}X_t, \frac{1}{n}\sum_{t=3}^{n}X_{t-2}X_t\Big) - \phi_1(\gamma_X(0),\gamma_X(1),\gamma_X(2)) \\ \phi_2\Big(\frac{1}{n}\sum_{t=1}^{n}X_t^2, \frac{1}{n}\sum_{t=2}^{n}X_{t-1}X_t, \frac{1}{n}\sum_{t=3}^{n}X_{t-2}X_t\Big) - \phi_2(\gamma_X(0),\gamma_X(1),\gamma_X(2)) \end{pmatrix}$$

$$\rightarrow_d N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \tau_1^2 & \tau_{12} \\ \tau_{21} & \tau_2^2 \end{pmatrix}\right),$$

where

$$\begin{aligned} \tau_1^2 &= Var(A_1Z_1 + A_2Z_2 + A_3Z_3) \\ &= A_1^2V_{0,0} + A_2^2V_{1,1} + A_3^2V_{2,2} + 2A_1A_2V_{0,1} + 2A_1A_3V_{0,2} + 2A_2A_3V_{1,2}, \\ \tau_2^2 &= Var(B_1Z_1 + B_2Z_2 + B_3Z_3) \\ &= B_1^2V_{0,0} + B_2^2V_{1,1} + B_3^2V_{2,2} + 2B_1B_2V_{0,1} + 2B_1B_3V_{0,2} + 2B_2B_3V_{1,2}, \\ \tau_{12} = \tau_{21} &= Cov(A_1Z_1 + A_2Z_2 + A_3Z_3, B_1Z_1 + B_2Z_2 + B_3Z_3). \end{aligned}$$

An analogous representation holds for the bootstrapped version (see, *e.g* [3], [9]). The residuals bootstrapping procedure used was proposed in [7] as follows. Define the residuals $\widehat{\varepsilon}_t = X_t - \Big(\widehat{\theta}_1X_{t-1} + \widehat{\theta}_2X_{t-2}\Big)$, $t = 3,4,\ldots,n$. The bootstrap sample $X_1^*, X_2^*, \ldots, X_n^*$ are obtained by resampling without replacement from the residuals $\widehat{\varepsilon}_3, \widehat{\varepsilon}_4, \ldots, \widehat{\varepsilon}_n$. Let $\widehat{F}_n$ be the empirical distribution of $\widehat{\varepsilon}_3, \widehat{\varepsilon}_4, \ldots, \widehat{\varepsilon}_n$, puts mass $1/n$ at each of the computed residuals. Now, the sequence of bootstrap residuals $\varepsilon_3^*, \varepsilon_4^*, \ldots, \varepsilon_n^*$ be conditionally independent with common distribution $\widehat{F}_n$. Given $X_j^* = X_j$, $j = 1,2$, as initial bootstrap sample, and we obtain $X_t^* = \widehat{\theta}_1X_{t-1} + \widehat{\theta}_2X_{t-2} + \varepsilon_t^*$, $t = 3,4,\ldots,n$. Both [3] and [7] proved that the residuals bootstrapping work well when it is applied to the autoregressive model.

We can see that the estimator $\widehat{\boldsymbol{\theta}}^* = (\widehat{\theta}_1^*, \widehat{\theta}_2^*)^T$ can be written as

$$\boldsymbol{\phi}\Big(\sum_{t=1}^{n}X_t^{2*}, \sum_{t=2}^{n}X_t^*X_{t-1}^*, \sum_{t=3}^{n}X_t^*X_{t-2}^*\Big)$$

for a measurable map $\boldsymbol{\phi} : \mathbf{R}^3 \rightarrow \mathbf{R}^2$,

$$\phi(u,v,w) = (\phi_1(u,v,w), \phi_2(u,v,w))^T,$$

with $\phi_1, \phi_2 : \mathbf{R}^3 \rightarrow \mathbf{R}$ be measurable functions as defined in (8). The function ϕ_1 is differentiable with derivative matrix

$$\phi'_{1(\widehat{\gamma}_X(0),\widehat{\gamma}_X(1),\widehat{\gamma}_X(2))} =$$

$$\begin{pmatrix} \frac{-\widehat{\gamma}_X(1)\big(\widehat{\gamma}_X(0)^2+\widehat{\gamma}_X(1)^2-2\widehat{\gamma}_X(0)\widehat{\gamma}_X(2)\big)}{\big(\widehat{\gamma}_X(0)^2-\widehat{\gamma}_X(1)^2\big)^2} & \frac{\big(\widehat{\gamma}_X(0)-\widehat{\gamma}_X(2)\big)\big(\widehat{\gamma}_X(0)^2+\widehat{\gamma}_X(1)^2\big)}{\big(\widehat{\gamma}_X(0)^2-\widehat{\gamma}_X(1)^2\big)^2} & \frac{-\widehat{\gamma}_X(1)}{\widehat{\gamma}_X(0)^2-\widehat{\gamma}_X(1)^2} \end{pmatrix}.$$

Also, the function ϕ_2 is differentiable with derivative matrix

$$\phi'_{2(\widehat{\gamma}_X(0),\widehat{\gamma}_X(1),\widehat{\gamma}_X(2))} =$$

$$\begin{pmatrix} \frac{2\widehat{\gamma}_X(0)\widehat{\gamma}_X(1)^2-\widehat{\gamma}_X(0)^2\widehat{\gamma}_X(2)-\widehat{\gamma}_X(1)^2\widehat{\gamma}_X(2)}{\big(\widehat{\gamma}_X(0)^2-\widehat{\gamma}_X(1)^2\big)^2} & \frac{2\widehat{\gamma}_X(0)\widehat{\gamma}_X(1)\big(\widehat{\gamma}_X(2)-\widehat{\gamma}_X(0)\big)}{\big(\widehat{\gamma}_X(0)^2-\widehat{\gamma}_X(1)^2\big)^2} & \frac{\widehat{\gamma}_X(0)}{\widehat{\gamma}_X(0)^2-\widehat{\gamma}_X(1)^2} \end{pmatrix}.$$

Now, we are ready to investigate the asymptotic distribution of the random vector $\widehat{\boldsymbol{\theta}}^* = (\widehat{\theta}_1^*, \widehat{\theta}_2^*)^T$, bootstrap version of $\widehat{\boldsymbol{\theta}} = (\widehat{\theta}_1, \widehat{\theta}_2)^T$. For shake the simplicity, let

$$\begin{aligned}
C_1 &= \frac{-\widehat{\gamma}_X(1)(\widehat{\gamma}_X(0)^2 + \widehat{\gamma}_X(1)^2 - 2\widehat{\gamma}_X(0)\widehat{\gamma}_X(2))}{(\widehat{\gamma}_X(0)^2 - \widehat{\gamma}_X(1)^2)^2}, \\
C_2 &= \frac{(\widehat{\gamma}_X(0) - \widehat{\gamma}_X(2))(\widehat{\gamma}_X(0)^2 + \widehat{\gamma}_X(1)^2)}{(\widehat{\gamma}_X(0)^2 - \widehat{\gamma}_X(1)^2)^2}, \\
C_3 &= \frac{-\widehat{\gamma}_X(1)}{\widehat{\gamma}_X(0)^2 - \widehat{\gamma}_X(1)^2}, \\
D_1 &= \frac{2\widehat{\gamma}_X(0)\widehat{\gamma}_X(1)^2 - \widehat{\gamma}_X(0)^2\widehat{\gamma}_X(2) - \widehat{\gamma}_X(1)^2\widehat{\gamma}_X(2)}{(\widehat{\gamma}_X(0)^2 - \widehat{\gamma}_X(1)^2)^2}, \\
D_2 &= \frac{2\widehat{\gamma}_X(0)\widehat{\gamma}_X(1)(\widehat{\gamma}_X(2) - \widehat{\gamma}_X(0))}{(\widehat{\gamma}_X(0)^2 - \widehat{\gamma}_X(1)^2)^2}, \\
D_3 &= \frac{\widehat{\gamma}_X(0)}{\widehat{\gamma}_X(0)^2 - \widehat{\gamma}_X(1)^2}.
\end{aligned}$$

By applying Theorem 3.1, we obtain

$$\begin{aligned}
&\sqrt{n}\Bigg(\boldsymbol{\phi}\Big(\tfrac{1}{n}\textstyle\sum_{t=1}^n X_t^{2*}, \tfrac{1}{n}\sum_{t=2}^n X_{t-1}^*X_t^*, \tfrac{1}{n}\sum_{t=3}^n X_{t-2}^*X_t^*\Big) - \boldsymbol{\phi}(\widehat{\gamma}_X(0), \widehat{\gamma}_X(1), \widehat{\gamma}_X(2))\Bigg) \\
&\qquad = \begin{pmatrix} \phi'_{1(\widehat{\gamma}_X(0),\widehat{\gamma}_X(1),\widehat{\gamma}_X(2))} \\ \phi'_{2(\widehat{\gamma}_X(0),\widehat{\gamma}_X(1),\widehat{\gamma}_X(2))} \end{pmatrix} \begin{pmatrix} \sqrt{n}\Big(\frac{1}{n}\sum_{t=1}^n X_t^{2*} - \widehat{\gamma}_X(0)\Big) \\ \sqrt{n}\Big(\frac{1}{n}\sum_{t=2}^n X_{t-1}^*X_t^* - \widehat{\gamma}_X(1)\Big) \\ \sqrt{n}\Big(\frac{1}{n}\sum_{t=3}^n X_{t-2}^*X_t^* - \widehat{\gamma}_X(2)\Big) \end{pmatrix} + \mathbf{o}_p(1) \\
&\qquad = \begin{pmatrix} C_1 & C_2 & C_3 \\ D_1 & D_2 & D_3 \end{pmatrix} \begin{pmatrix} \sqrt{n}\Big(\frac{1}{n}\sum_{t=1}^n X_t^{2*} - \widehat{\gamma}_X(0)\Big) \\ \sqrt{n}\Big(\frac{1}{n}\sum_{t=2}^n X_{t-1}^*X_t^* - \widehat{\gamma}_X(1)\Big) \\ \sqrt{n}\Big(\frac{1}{n}\sum_{t=3}^n X_{t-2}^*X_t^* - \widehat{\gamma}_X(2)\Big) \end{pmatrix} + \mathbf{o}_p(1).
\end{aligned}$$

According to Theorem 2.2, the multivariate limiting distribution of random variables $\Big(\frac{1}{n}\sum_{t=1}^n X_t^{2*}, \frac{1}{n}\sum_{t=2}^n X_t^*X_{t-1}^*, \frac{1}{n}\sum_{t=3}^n X_t^*X_{t-2}^*\Big)^T$ is

$$\begin{aligned}
&\sqrt{n}\left(\begin{pmatrix} \frac{1}{n}\sum_{t=1}^n X_t^{2*} \\ \frac{1}{n}\sum_{t=2}^n X_t^*X_{t-1}^* \\ \frac{1}{n}\sum_{t=3}^n X_t^*X_{t-2}^* \end{pmatrix} - \begin{pmatrix} \widehat{\gamma}_X(0) \\ \widehat{\gamma}_X(1) \\ \widehat{\gamma}_X(2) \end{pmatrix}\right) \\
&\qquad \to_d N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} V_{0,0}^* & V_{0,1}^* & V_{0,2}^* \\ V_{1,0}^* & V_{1,1}^* & V_{1,2}^* \\ V_{2,0}^* & V_{2,1}^* & V_{2,2}^* \end{pmatrix}\right). \qquad (11)
\end{aligned}$$

Meanwhile, by Theorem 3.1, if $(W_1, W_2, W_3)^T$ posses multivariate normal distribution as in (11), then

$$\begin{pmatrix} C_1 & C_2 & C_3 \\ D_1 & D_2 & D_3 \end{pmatrix} \begin{pmatrix} \sqrt{n}\Big(\frac{1}{n}\sum_{t=1}^n X_t^{2*} - \widehat{\gamma}_X(0)\Big) \\ \sqrt{n}\Big(\frac{1}{n}\sum_{t=2}^n X_{t-1}^*X_t^* - \widehat{\gamma}_X(1)\Big) \\ \sqrt{n}\Big(\frac{1}{n}\sum_{t=3}^n X_{t-2}^*X_t^* - \widehat{\gamma}_X(2)\Big) \end{pmatrix}$$

$$\rightarrow_d \begin{pmatrix} C_1 & C_2 & C_3 \\ D_1 & D_2 & D_3 \end{pmatrix} \begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix} \sim N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \tau_1^{2*} & \tau_{12}^* \\ \tau_{21}^* & \tau_2^{2*} \end{pmatrix} \right),$$

where $\tau_1^{2*}, \tau_2^{2*}, \tau_{12}^*$ dan τ_{21}^* are bootstrap version of $\tau_1^2, \tau_2^2, \tau_{12}$ dan τ_{21} respectively. Hence, by Theorem 3.2, we conclude that

$$\sqrt{n}(\widehat{\boldsymbol{\theta}}^* - \widehat{\boldsymbol{\theta}}) \rightarrow_d N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \tau_1^{2*} & \tau_{12}^* \\ \tau_{21}^* & \tau_2^{2*} \end{pmatrix} \right).$$

5. Conclusions

We conclude that the bootstrap parameter estimator of the AR(2) process is asymptotic and converge in distribution to the bivariate normal distribution.

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