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NATURAL EXTENSIONS AND ENTROPY OF α-CONTINUED FRACTION EXPANSION MAPS WITH ODD PARTIAL QUOTIENTS

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Abstract. In [1], Boca and the fourth author of this paper introduced a new class of continued fraction expansions *with odd partial quotients*, parameterized by a parameter $\alpha \in [g, G]$, where $g = \frac{1}{2}(\sqrt{5}-1)$ and $G = g+1 = 1/g$ are the two *golden mean numbers*. Using operations called *singularizations* and *insertions* on the partial quotients of the odd continued fraction expansions under consideration, the natural extensions from [1] are obtained, and it is shown that for each $\alpha, \alpha^* \in [g, G]$ the natural extensions from [1] are metrically isomorphic. An immediate consequence of this is, that the entropy of all these natural extensions is equal for $\alpha \in [g, G]$, a fact already observed in [1]. Furthermore, it is shown that this approach can be extended to values of α smaller than g, and that for values of $\alpha \in [\frac{1}{6}(\sqrt{13}-1), g]$ all natural extensions are still isomorphic. In the final section of this paper further attention is given to the entropy, as function of $\alpha \in [0, G]$. It is shown that if there exists an ergodic, absolutely continuous T_α-invariant measure, in any neighborhood of 0 we can find intervals on which the entropy is decreasing, intervals on which the entropy is increasing and intervals on which the entropy is constant. Moreover, we identify the largest interval on which the entropy is constant. In order to prove this we use a phenomenon called matching.

1. **Introduction.** It is well known that every real number x can be written as a (regular) continued fraction, of the form

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n + \ddots}}} = [a_0; a_1, a_2, \ldots, a_n, \ldots], \tag{1}$$

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where $a_0 \in \mathbb{Z}$ such that $x - a_0 \in [0,1)$, and $a_n \in \mathbb{N}$ for $n \geq 1$. Such a *regular continued fraction expansion* (RCF) of x is infinite and unique if and only if x is irrational; in case $x \in \mathbb{Q}$ one has two finite expansions of the form (1).

There are many continued fraction algorithms in addition to the regular continued fractions, such as Nakada's α-expansions from [20, 14, 18, 21] and the Ito-Tanaka α-expansions from [29] which generalize the RCF, the Rosen fractions (see [24, 2, 15]) which generalize the *nearest integer continued fraction*, and the continued fractions with either *odd* or *even* partial quotients (see [25, 26, 22, 27, 28, 9]).

Recently, Boca and the fourth author introduced in [1] a new class of continued fraction expansions: α-continued fraction expansions with odd partial quotients. These α-expansions are reminiscent of Nakada's α-expansions from [20] and are studied for a certain range of the parameter. Nakada's α-continued fractions were first studied in [20] for $\alpha \in [\frac{1}{2}, 1]$, which was extended to $[\sqrt{2}-1, 1]$ in [19] and later in [18] to $[0,1]$. Boca and the fourth author studied the α-Odd expansions for $\alpha \in [g, G]$, where $g = (\sqrt{5}-1)/2$ and $G = g + 1 = 1/g = (\sqrt{5}+1)/2$ are the two *golden mean* numbers. For both types of α-expansions, the authors show that the underlying dynamical system is ergodic, find its natural extension and obtain the entropy for each α under consideration. The natural extension is an invertible measure preserving dynamical system such that the original system is a factor and the structure of the σ-algebra is obtained by "lifting" the original σ-algebra, see for example [8], page 99 or [7] page 72 for a formal definition. As one of the referees remarked, the idea of a natural extension was originally introduced as an inverse limit by Rohlin in 1961; c.f. [23]. For $\alpha = 0.9g$, Boca and the fourth author use a simulation to estimate the domain of the natural extension of this α-expansion, which is quite different from the case where $\alpha \in [g, G]$.

In this paper, we obtain the results from [1] in a very different way, enabling us to also obtain the underlying dynamical system for $\alpha \in [\frac{\sqrt{13}-1}{6}, g)$. In particular, our construction yields that all natural extensions under consideration are metrically isomorphic, therefore confirming the result of Boca and the fourth author for $\alpha \in [g, G]$. We would like to remark that for $\alpha \leq 1$, we do not have that all branches are expansive. In [25], Schweiger proved for $\alpha = 1$ that the corresponding system is ergodic. By using the isomorphism we construct, it follows that for all $\alpha \in [\frac{\sqrt{13}-1}{6}, G]$, the corresponding dynamical system is ergodic.

In this article we prove the following theorem:

Theorem 1.1. *Let $\frac{\sqrt{13}-1}{6} \leq \alpha < G$, then the domain of a version of the natural extension, Ω_α of the odd α-continued fraction expansion map T_α from (3) can be explicitely given. Furthermore, the dynamical system $(\Omega_\alpha, \mathcal{B}_\alpha, \mu_\alpha, \mathcal{T}_\alpha)$, where $\mathcal{B}_\alpha$ is the collection of Borel subsets of Ω_α, and μ_α is a probability measure on $(\Omega_\alpha, \mathcal{B}_\alpha)$ with density*

$$d_\alpha(x,y) = \frac{1}{3\log G}\frac{1}{(1+xy)^2}, \quad \textit{for } (x,y) \in \Omega_\alpha,$$

and $d_\alpha(x,y) = 0$ elsewhere, is ergodic and metrically isomorphic to the natural extension for any other $\alpha^* \in [\frac{\sqrt{13}-1}{6}, G]$. As a consequence, we have that the Kolmogorov-Sinai entropy equals $\pi^2/(9\log G)$ for these values of α.

This theorem is proved for $\alpha \in (g, 1]$ in Section 2.2, for $\alpha \in [1, G)$ in Section 2.3, and for $\alpha \in [\frac{\sqrt{13}-1}{6}, g)$ in Section 3. The domains are explicitely given in the correponding sections, but require too much notation to state here. Note that the difficulty lies in finding the right domain. From the proofs, it will follow that the dynamical systems are isomorphic. The other results in the theorem follow immediately. In the second part of the article, Section 4, we extend the parameter space to $[0, G]$ and study matching and entropy. In particular, we prove the following two theorems:

Theorem 1.2. *There exists sequences of intervals $(I_n), (J_n), (K_n), (L_n), n \geq 3$ such that*

1. $\frac{1}{n} \in I_n$,
2. $I_{n+1} < J_n < K_n < L_n < I_n$,
3. *The entropy of T_α is increasing on I_n, decreasing on K_n and constant on J_n and L_n for every $n \geq 3$.*

Here $I < J$ means that any element of J is strictly larger than any element of I.

This is analogous to a result on α-continued fractions in a paper by Nakada and Natsui [21]. Entropy is often studied together with matching (it will become clear in Section 4 why this is the case). We say that matching holds for parameter α if there are $N, M \in \mathbb{N}$ such that

$$T_\alpha^N(\alpha) = T_\alpha^M(\alpha - 2). \tag{2}$$

We call the minimal N and M for which (2) holds the *matching exponents*. An interval for which all parameters have the same matching exponents is called a *matching interval*. The terminology 'matching', 'matching exponents', and 'matching interval' was introduced in [6]. In Section 4 we also prove the following theorem.

Theorem 1.3. *The intervals $([0; 2, \overline{1^{2n}, 3}], [0; \overline{2, 1^{2n-2}, 2}])$ are matching intervals with matching exponents $(n+3, n+3)$. Furthermore, the left convergents $l_{2n+1} = [0; 2, 1^{2n+1}]$ of g^2 match with matching exponents $(n+1, n+4)$.*

Here with $[0; \overline{2, 1^{2n}, 3}]$ we mean the periodic regular continued fraction with digit sequence $2, 1, 1, \ldots, 1, 1, 3, 2, 1, 1, \ldots, 1, 1, 3, 2, 1, 1, \ldots, 1, 1, 3 \ldots$. Note that this implies that the largest entropy plateaux is contained in $[g^2, G]$.

2. Background.

Let $\alpha \in [0, G]$, and set $I_\alpha = [\alpha - 2, \alpha)$. Then for $x \in I_\alpha$, the α-continued fraction map $T_\alpha : I_\alpha \to I_\alpha$ is defined in [1] as

$$T_\alpha(x) = \frac{\varepsilon(x)}{x} - d_\alpha(x), \quad \text{if } x \in I_\alpha \setminus \{0\}, \tag{3}$$

and $T_\alpha(0) = 0$, where $\varepsilon(x) = \text{sign}(x)$ and

$$d_\alpha(x) = 2\left\lfloor \frac{1}{2|x|} + \frac{1-\alpha}{2} \right\rfloor + 1, \quad \text{if } x \in I_\alpha \setminus \{0\}.$$

See Figure 1 for an example.

For $x \in I_\alpha \setminus \{0\}$, the map T_α "generates" a continued fraction in the following way: if $T_\alpha^{n-1}(x) \neq 0$ for $n \geq 1$, set $\varepsilon_n = \text{sign}(T_\alpha^{n-1}(x))$ and $a_n = d_\alpha(T_\alpha^{n-1}(x))$. Note that $T_\alpha^{n-1}(x) = 0$ for some $n \geq 1$ if and only if $x \in \mathbb{Q}$. Then from (3) it follows that:

$$x = \frac{\varepsilon_1}{a_1 + T_\alpha(x)}. \tag{4}$$

Since in general for $n \geq 1$,

$$T_\alpha^n(x) = \frac{\varepsilon_n}{T_\alpha^{n-1}(x)} - a_n \tag{5}$$

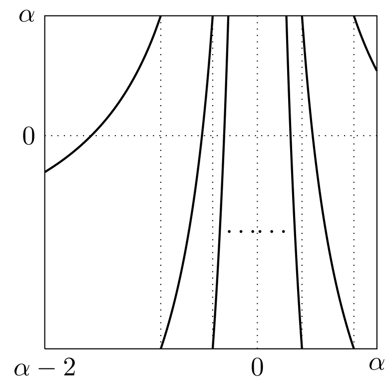

FIGURE 1. The map T_α for $\alpha = 0.72$.

we find that

$$T_\alpha^{n-1}(x) = \frac{\varepsilon_n}{a_n + T_\alpha^n(x)}. \tag{6}$$

Then from (4) and (6), we find that, if $T_\alpha^{n-1}(x) \neq 0$,

$$x = \cfrac{\varepsilon_1}{a_1 + \cfrac{\varepsilon_2}{a_2 + T_\alpha^2(x)}} = \cdots = \cfrac{\varepsilon_1}{a_1 + \cfrac{\varepsilon_2}{a_2 + \ddots + \cfrac{\varepsilon_n}{a_n + T_\alpha^n(x)}}}. \tag{7}$$

If x is rational, there exists an $n \in \mathbb{N}$ such that $T_\alpha^n(x) = 0$, and thus the expansion of x in (7) is finite. If x is irrational it follows from (3) that $T_\alpha^n(x) \neq 0$ for all $n \geq 1$. In this case, deleting $T_\alpha^n(x)$ from (7) yields a so-called *convergent*:

$$\frac{p_{\alpha,n}}{q_{\alpha,n}} = \cfrac{\varepsilon_1}{a_1 + \cfrac{\varepsilon_2}{a_2 + \ddots + \cfrac{\varepsilon_n}{a_n}}}, \tag{8}$$

where we assume that $p_{\alpha,n}$ and $q_{\alpha,n} > 0$ are relatively prime integers. Similar to the regular continued fraction (see e.g. [8, 10]), we obtain[1] that

$$x = \lim_{n\to\infty} \frac{p_{\alpha,n}}{q_{\alpha,n}}.$$

In view of this we write

$$x = \cfrac{\varepsilon_1}{a_1 + \cfrac{\varepsilon_2}{a_2 + \ddots + \cfrac{\varepsilon_n}{a_n + \ddots}}},$$

which is denoted as $x = [0; \varepsilon_1/a_1, \varepsilon_2/a_2, \dots, \varepsilon_n/a_n, \dots]$.[2] In later sections we will use that the $p_{\alpha,n}$ and $q_{\alpha,n}$ satisfy[3] recurrence relations given by:

$$p_{\alpha,-1} := 1 \quad p_{\alpha,0} := 0 \quad p_{\alpha,n} = a_n p_{\alpha,n-1} + \varepsilon_n p_{\alpha,n-2} \tag{9}$$

[1] Convergence of the continued fractions can be found in [1] for $\alpha \in [g, G]$, follows from our results for $\alpha \in [\frac{\sqrt{13}-1}{6}, G]$ and is assumed in Section 4 for $\alpha \in [0, \frac{\sqrt{13}-1}{6})$.

[2] In Section 4 we use the notation $x = [0; \varepsilon_1 a_1, \varepsilon_2 a_2, \dots, \varepsilon_n a_n, \dots]$ in order to save space. Since for $i \geq 1$ the partial quotients are odd positive integers the meaning is still straightforward.

[3] When there is no confusion possible we suppress the α in $p_{\alpha,n}$ and $q_{\alpha,n}$.

$$q_{\alpha,-1} := 0 \quad q_{\alpha,0} := 1 \quad q_{\alpha,n} = a_n q_{\alpha,n-1} + \varepsilon_n q_{\alpha,n-2}. \tag{10}$$

The case $\alpha = 1$, which will be the starting case of our investigations, has previously been studied by Schweiger in [25, 26] and Rieger in [22]. In particular, Schweiger obtained in [25] the natural extension of the *continued fraction with odd partial quotients* (oddCF). This natural extension is defined on $[0,1) \times [-g^2, G]$, and is – as we will see shortly – an isomorphic copy of the system found by Boca and the fourth author. It is nice to note, that Schweiger's natural extension has Rieger's *grotesque continued fraction* (GCF) as the inverse of the second coordinate map. That is, Rieger's GCF are the dual continued fraction expansions of Schweiger's oddCF, which is the case $\alpha = G$ in [1]. In fact, if we would consider in Schweiger's natural extension the inverse of his map, and exchange the order of the coordinates, we get the system for $\alpha = G$ as obtained in [1]. Sebe also obtained the natural extension of the GCF; see [28]. We will also use these systems as starting points of our investigations; see Subsection 2.3.

2.1. **The continued fraction expansion with odd partial quotients.** The case $\alpha = 1$ corresponds to Schweiger's *continued fractions with odd partial quotients*. In [26], Schweiger defines his continued fraction map $T : [0,1) \to [0,1]$ as follows: let

$$B(+1,k) = \left(\frac{1}{2k}, \frac{1}{2k-1}\right], \quad \text{for } k = 1, 2, \dots,$$

and

$$B(-1,k) = \left(\frac{1}{2k-1}, \frac{1}{2k-2}\right], \quad \text{for } k = 2, 3, \dots,$$

then the map

$$T(x) = \varepsilon\left(\frac{1}{x} - (2k-1)\right), \quad \text{for } x \in B(\varepsilon, k) \text{ and } \varepsilon = \pm 1,$$

yields the oddCF-expansion of x. Schweiger showed that the map $\mathcal{T} : [0,1] \times [-g^2, G] \to [0,1] \times [-g^2, G]$, defined by

$$\mathcal{T}(x,y) = \left(T(x), \frac{\varepsilon}{a+y}\right),$$

where $\varepsilon = \pm 1$ and $a = 2k-1$ are such that $x \in B(\varepsilon, k)$, is the natural extension map of T, and that the system

$$(\Omega, \mathcal{B}, \bar{\mu}, \mathcal{T}) \tag{11}$$

is an ergodic system. Here $\Omega = [0,1] \times [-g^2, G]$, $\mathcal{B}$ is the collection of Borel sets of Ω, and $\bar{\mu}$ is a $\mathcal{T}$-invariant probability measure on Ω with density $(3 \log G)^{-1}(1+xy)^{-2}$. Now define the map $M : \Omega \to \mathbb{R}^2$ by

$$M(x,y) = \begin{cases} (x,y), & \text{if } y \geq 0; \\ (-x,-y), & \text{if } y < 0. \end{cases}$$

Now set $\Omega_1 = M(\Omega)$, and let $\mathcal{T}_1 : \Omega_1 \to \Omega_1$ be defined as:

$$\mathcal{T}_1(x,y) = M(\mathcal{T}(M^{-1}(x,y))), \quad \text{for } (x,y) \in \Omega_1. \tag{12}$$

Since M preserves the $\bar{\mu}$ measure, and since

$$\mathcal{T}_1(x,y) = \left(T_1(x), \frac{1}{d_1(x) + \varepsilon(x)y}\right), \tag{13}$$

we see that the dynamical system

$$(\Omega_1, \mathcal{B}_1, \bar{\mu}_1, \mathcal{T}_1)$$

is metrically isomorphic with Schweiger's system $(\Omega, \mathcal{B}, \bar{\mu}, \mathcal{T})$ from (11). It is exactly this ergodic system which was obtained in [1] for the case $\alpha = 1$. Note that $\mathcal{B}_1$ is the collection of Borel subsets of Ω_1, and that on Ω_1 the $\mathcal{T}_1$-invariant probability measure $\bar{\mu}_1$ has density $(3 \log G)^{-1}(1 + xy)^{-2}$. For the remainder of this section we work with this dynamical system. See Figure 2 for both planar regions Ω and Ω_1.

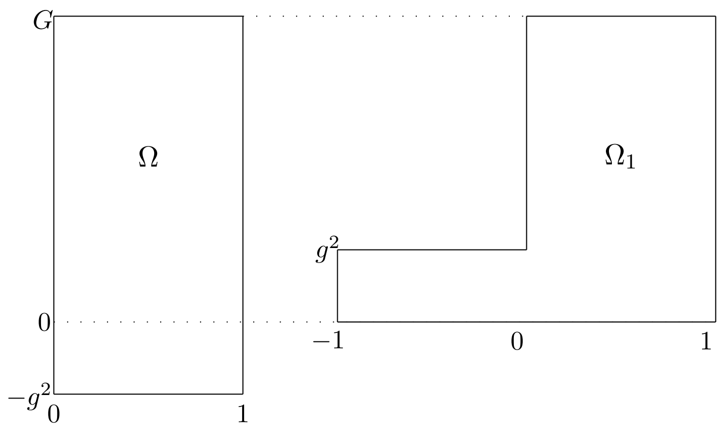

FIGURE 2. The domains Ω and Ω_1.

Later in this section, we will show how the planar natural extensions found by Boca and the fourth author for $\alpha \in [g, 1)$ can be obtained from Ω_1 via some simple operations on the partial quotients of any $x \in I_1 \setminus \{0\}$ (see Subsection 2.2), while we will deal with the case $\alpha \in (1, G]$ in Subsection 2.3. In Section 3 we will find the planar natural extension in case $\alpha \in [\frac{\sqrt{13}-1}{6}, g]$ which is the more difficult case.

Although we still need to determine the domain Ω_α, the natural extension map $\mathcal{T}_\alpha : \Omega_\alpha \to \Omega_\alpha$ is for each α given by:

$$\mathcal{T}_\alpha(x, y) = \left(T_\alpha(x), \frac{1}{d_\alpha(x) + \varepsilon(x)y} \right); \tag{14}$$

see (13) for the case $\alpha = 1$. For $x \in I_\alpha \setminus \{0\}$, and for $n \geq 1$ for which $T_\alpha^n(x) \neq 0$ (so this is for all $n \geq 1$ for all irrational $x \in I_\alpha$) we set:

$$(t_n, v_n) = \mathcal{T}_\alpha^n(x, 0).$$

Now if the α-expansion of x is given by $x = [0; \varepsilon_1/a_1, \dots, \varepsilon_n/a_n, \varepsilon_{n+1}/a_{n+1}, \dots]$, then

$$t_n = T_\alpha^n(x) = [0; \varepsilon_{n+1}/a_{n+1}, \varepsilon_{n+2}/a_{n+2}, \dots],$$

and using induction and the recurrence relations (9) and (10),

$$v_n = [0; 1/a_n, \varepsilon_n/a_{n-1}, \varepsilon_{n-1}/a_{n-2}, \dots, \varepsilon_2/a_1] = \frac{q_{n-1}}{q_n}.$$

So t_n is the *future* of x at time n, and v_n is the *past* of x at time n in the α-expansion of x. Of course, for different α the future and past of x at time n might be different.

For details and proofs, please consult [8, 10]. In the next section, we will show how to obtain Ω_α from Ω_1, using singularizations and insertions. Our construction will show that all these systems are (metrically) isomorphic.

2.2. **Singularizations and insertions, part 1.** Here we will prove the following theorem.

Theorem 2.1. *Let $\alpha \in [g,1)$ and $x \in [\alpha-2,\alpha)$. The α-continued fraction expansion with odd partial quotients of x can be found by using singularizations and insertions. Via this method we also obtain the domain of the natural extension explicitly. Furthermore, we find that the dynamical system is metrically isomorphic to the system for $\alpha = 1$.*

The description of the domain of the natural extension is given explicitly at the end of the proof. It is also worth mentioning that since the dynamical systems for all $\alpha \in [g,1]$ are isomorphic, the systems will have – among other things – the same entropy.

Proof. The following identities will be frequently used:

Singularization. Let $A, B \in \mathbb{Z}$ with $A \geq 0$ and $B \geq 1$ and let $\xi > -1$, then:

$$A + \cfrac{1}{1 + \cfrac{1}{B+\xi}} = A + 1 + \frac{-1}{B+1+\xi}.$$

Insertion. Let $A, B \in \mathbb{N}$, $B \geq 3$, and let $\xi > -1$, then:

$$A + \frac{-1}{B+\xi} = A + 1 + \cfrac{-1}{1 + \cfrac{-1}{B+1+\xi}}.$$

The proofs of these two identities are left to the reader. In Subsection 2.3 and Section 3 we will see variations of these singularization and insertion identities.

Now let $x \in I_1 = [-1,1)$, and suppose that $n \geq 0$ is the first "time" that $T_1^n(x) \geq \alpha$ (note that $n = 0$ is allowed; n is the first "time" that $T_1^n(x) \notin I_\alpha = [\alpha-1,\alpha)$). Since $g \leq \alpha < 1$ and since the time-1 cylinder $\Delta(+1,1)$ for the T_1-map is given by

$$\Delta(+1,1) = \{x \in I_1 \,|\, \varepsilon(x) = +1,\, d_1(x) = 1\} = \left(\tfrac{1}{2}, 1\right),$$

the T_1-expansion of x is given by:

$$x = \cfrac{\varepsilon_1}{a_1 + \ddots + \cfrac{\varepsilon_n}{a_n + \cfrac{1}{1 + \cfrac{1}{a_{n+2} + \cfrac{\varepsilon_{n+3}}{a_{n+3} + \ddots}}}}}. \tag{15}$$

Here $\varepsilon_n = \varepsilon(T_1^{n-1}(x))$ and $a_n = d_1(T_1^{n-1}(x))$, for $n \geq 1$ (note that the expansion of x in (15) is finite if and only if $x \in \mathbb{Q}$). So if $T_1^n(x) \geq \alpha$ we have that $a_{n+1} = \varepsilon_{n+1} = \varepsilon_{n+2} = 1$. The expansion from (15) is denoted as

$$x = [0;\, \varepsilon_1/a_1, \ldots, \varepsilon_n/a_n, 1/1, 1/a_{n+2}, \varepsilon_{n+3}/a_{n+3}, \ldots].$$

Now singularizing $a_{n+1} = 1$ in (15) yields the following continued fraction expansion of x:

$$x = \cfrac{\varepsilon_1}{a_1 + \ddots + \cfrac{\varepsilon_n}{a_n + 1 + \cfrac{-1}{a_{n+2} + 1 + \cfrac{\varepsilon_{n+3}}{a_{n+3} + \ddots}}}}. \tag{16}$$

In (16) two of the partial quotients are *even* (viz. $a_n + 1$ and $a_{n+2} + 1$), so this expansion of x is **not** a continued fraction expansion of x with (only!) odd partial quotients. Now inserting in (16) $-1/1$ "between" $a_n + 1$ and $\frac{-1}{a_{n+2}+1+\dots}$ yields the following continued fraction expansion of x with (only) odd partial quotients:

$$x = \cfrac{\varepsilon_1}{a_1 + \ddots + \cfrac{\varepsilon_n}{a_n + 2 + \cfrac{-1}{1 + \cfrac{-1}{a_{n+2} + 2 + \cfrac{\varepsilon_{n+3}}{a_{n+3} + \ddots}}}}}; \tag{17}$$

which we denote by $x = [a_0^*; \varepsilon_1^*/a_1^*, \varepsilon_2^*/a_2^*, \dots]$; here $a_0^* = -1$ if $x \in [\alpha, 1)$, otherwise it is 0. Due to our construction we have that $\varepsilon_1^* = \varepsilon_1, \dots, \varepsilon_n^* = \varepsilon_n$, $\varepsilon_{n+1}^* = -1 = \varepsilon_{n+2}^*$, and that $\varepsilon_{n+k}^* = \varepsilon_{n+k}$, for $k \geq 3$. Similarly, for the partial quotients we have that $a_1^* = a_1, \dots, a_{n-1}^* = a_{n-1}$, $a_n^* = a_n + 2$, $a_{n+1}^* = 1$, $a_{n+2}^* = a_{n+2} + 2$, and that $a_{n+k}^* = a_{n+k}$, for $k \geq 3$. Setting

$$t_i^* = \cfrac{\varepsilon_{i+1}^*}{a_{i+1}^* + \ddots} = [0; \varepsilon_{i+1}^*/a_{i+1}^*, \dots],$$

and

$$v_i^* = \cfrac{1}{a_i^* + \cfrac{\varepsilon_i^*}{a_{i-1}^* + \ddots + \cfrac{\varepsilon_2^*}{a_1^*}}} = [0; 1/a_i^*, \varepsilon_i^*/a_{i-1}^*, \dots, \varepsilon_2^*/a_1^*],$$

we see that

$$(t_i^*, v_i^*) = (t_i, v_i) \quad \text{for } i = 1, 2, \dots, n-1 \text{ and } i \geq n+2,$$

and that (t_n, v_n) and (t_{n+1}, v_{n+1}) got replaced by (t_n^*, v_n^*) respectively (t_{n+1}^*, v_{n+1}^*).

Since $(t_n, v_n) \in D_\alpha = [\alpha, 1) \times [0, G]$ we immediately have that

$$(t_{n+1}, v_{n+1}) \in H_\alpha := \mathcal{T}_1(D_\alpha) = \Big(0, \frac{1-\alpha}{\alpha}\Big] \times \Big[g^2, 1\Big].$$

For the new continued fraction expansion (17) of x these rectangles D_α and H_α have been "vacated"; see also Figure 3.

Where do (t_n^*, v_n^*) and (t_{n+1}^*, v_{n+1}^*) "live"? We see from (15) that the time n future t_n in the α-expansion of x, for $\alpha = 1$ is given by

$$t_n = \cfrac{1}{1 + \cfrac{1}{a_{n+2} + \cfrac{\varepsilon_{n+3}}{a_{n+3} + \ddots}}},$$

while for the "new" time n future t_n^* of x in (17) we have:

$$t_n^* = \cfrac{-1}{1+\cfrac{-1}{a_{n+2}+2+\ddots}}.$$

Since

$$\begin{aligned} t_n &= \cfrac{1}{1+\cfrac{1}{a_{n+2}+\ddots}} = 1+\cfrac{-1}{a_{n+2}+1+\ddots} \\ &= 1+1+\cfrac{-1}{1+\cfrac{-1}{a_{n+2}+1+1+\ddots}} = 2+t_n^*, \end{aligned}$$

we see that

$$t_n^* = t_n - 2. \tag{18}$$

Note that this is something we could expect beforehand from (5); since $t_n \in [\alpha, 1)$ we must increase the digit a_n by 2 to get that $t_n^* \in I_\alpha$.

For the time n past v_n in the α-expansion of x, for $\alpha = 1$ we have that

$$v_n = \frac{q_{n-1}}{q_n} = [0; 1/a_n, \varepsilon_n/a_{n-1}, \dots, \varepsilon_2/a_1],$$

while we see from (17), and also (10) that the "new" past v_n^* satisfies

$$\begin{aligned} v_n^* &= [0; 1/a_n+2, \varepsilon_n/a_{n-1}, \dots, \varepsilon_2/a_1] = \frac{1}{a_n+2+\varepsilon_n v_{n-1}} \\ &= \frac{1}{a_n+2+\varepsilon_n \frac{q_{n-2}}{q_{n-1}}} = \frac{q_{n-1}}{2q_{n-1}+a_nq_{n-1}+\varepsilon_n q_{n-2}} \\ &= \frac{q_{n-1}}{2q_{n-1}+q_n} = \frac{\frac{q_{n-1}}{q_n}}{2\frac{q_{n-1}}{q_n}+1} = \frac{v_n}{2v_n+1}. \end{aligned}$$

In view of this and (18) we define a map $M_\alpha : \Omega_1 \to \mathbb{R}^2$ as:

$$M_\alpha(t,v) = \begin{cases} (t,v), & \text{if } (t,v) \in \Omega_1 \setminus D_\alpha, \\ \left(t-2, \dfrac{v}{1+2v}\right), & \text{if } (t,v) \in D_\alpha. \end{cases} \tag{19}$$

Since $\alpha \in [g, 1)$, we have that $\frac{1}{\alpha} - 1 \le \alpha$, and that equality holds when $\alpha = g$. For $\alpha < g$ something special happens; see Section 3.

So we found that

$$(t_n, v_n) \in D_\alpha \quad \text{if and only if} \quad (t_n^*, v_n^*) \in M_\alpha(D_\alpha) = [\alpha-2, -1) \times \left[0, g^2\right].$$

A calculation yields that the probability measure $\bar{\mu}_1$ on Ω_1 is invariant under the map M_α.

Next we will determine where (t_{n+1}^*, v_{n+1}^*) lives. From (17) we see that

$$t_n^* = \frac{-1}{1+t_{n+1}^*},$$

implying that

$$t^*_{n+1} = \frac{-1}{t^*_n} - 1.$$

Now for $t \in [\alpha - 2, -1)$ we have that

$$T_\alpha(t) = \frac{-1}{t} - 1, \tag{20}$$

so we see that $t^*_{n+1} = T_\alpha(t^*_n)$. Furthermore, from (17) we see that

$$v^*_{n+1} = [0; 1/1, -1/a_n + 2, \varepsilon_n/a_{n-1}, \dots, \varepsilon_2/a_1]$$

and that

$$v^*_n = [0; 1/a_n + 2, \varepsilon_n/a_{n-1}, \dots, \varepsilon_2/a_1],$$

from which we see that

$$v^*_{n+1} = \frac{1}{1 - v^*_n}. \tag{21}$$

From (20) and (21) we thus find that

$$(t^*_{n+1}, v^*_{n+1}) = \mathcal{T}_\alpha(t^*_n, v^*_n).$$

Defining $M_\alpha : [\alpha - 2, -1) \times [0, g^2] \to \mathbb{R}^2$ by

$$M_\alpha(t, v) = \left(\frac{-1}{t} - 1, \frac{1}{1 - v} \right) (= \mathcal{T}_\alpha((t, v))),$$

and setting

$$\Omega_\alpha = (\Omega_1 \setminus (D_\alpha \cup \mathcal{T}_1(D_\alpha)) \cup M_\alpha(D_\alpha) \cup M^2_\alpha(D_\alpha),$$

(cf. Figure 3), it follows from the fact that for $k \geq 2$ we have that $(t^*_{n+k}, v^*_{n+k}) = (t_{n+k}, v_{n+k})$ we must have that

$$\mathcal{T}_\alpha \left(M^2_\alpha(D_\alpha) \right) = \mathcal{T}_\alpha \left(H_\alpha \right);$$

the "new material" $M^2_\alpha(D_\alpha)$ fills the hole in Ω_α caused by the "hole" H_α. The map $\mathcal{T}_\alpha$ is bijective on Ω_α, apart from a set of Lebesgue measure 0. We find that the dynamical system

$$(\Omega_\alpha, \mathcal{B}_\alpha, \bar{\mu}_\alpha, \mathcal{T}_\alpha)$$

is metrically isomorphic to the dynamical system

$$(\Omega_1, \mathcal{B}_1, \bar{\mu}_1, \mathcal{T}_1)$$

and therefore inherits all its ergodic properties. In particular, since the map M_α preserves the $\bar{\mu}_1$-meaure, we find that on Ω_α the $\mathcal{T}_\alpha$-invariant measure $\bar{\mu}_\alpha$ has density

$$\frac{1}{3 \log G} \frac{1}{(1 + xy)^2}$$

on Ω_α. Furthermore, since $(\Omega_1, \mathcal{B}_1, \bar{\mu}_1, \mathcal{T}_1)$ is (a version of) the natural extension of T_1 (cf. [25, 26, 22]), we find that $(\Omega_\alpha, \mathcal{B}_\alpha, \bar{\mu}_\alpha, \mathcal{T}_\alpha)$ is the natural extension of T_α for $g \leq \alpha < 1$; this is the system obtained by Boca and the fourth author in [1]. This concludes the proof of Theorem 2.1. □

Note that for $\alpha \in [g, 1)$ the map T_α is not expansive on part of its domain (on $[\alpha - 2, 1)$ the derivative of T_α is less than one). However, due to the isomorphism we constructed, the natural extension for T_α is still ergodic, yielding that T_α is *eventually expanding* and ergodic.

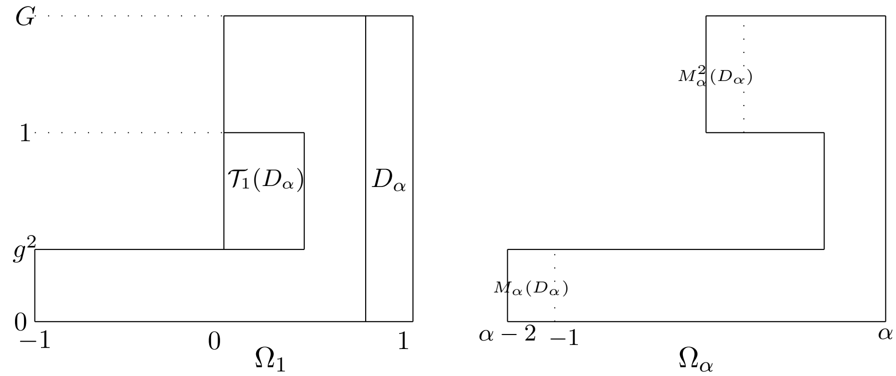

FIGURE 3. The domains Ω_1 and Ω_α for $\alpha \in [g, 1)$.

2.3. **Singularizations and insertions, part 2.** We will now turn to the case of $\alpha \in (1, G]$ and prove the following theorem which extends the domain of Theorem 2.1

Theorem 2.2. *Let $\alpha \in (1, G]$ and $x \in [\alpha - 2, \alpha)$. The α-continued fraction expansion with odd partial quotients of x can be found by using singularizations and insertions. Via this method we also obtain the domain of the natural extension explicitly. Furthermore, we find that the dynamical system is metrically isomorphic to the system for $\alpha = 1$.*

We could again start from the Schweiger-Rieger natural extension, and use appropriate singularizations and insertions to find the systems obtained in [1] for these values of α. In view of what we will see in Section 3 we decided to start with the natural extension of the *grotesque continued fraction expansion*. This dynamical system, which is case $\alpha = G$ in [1] is already present in the papers by Schweiger (cf. [25, 26]) and Rieger ([22]). So our starting point is the planar natural extension $\Omega_G = [-g^2, G) \times [0, 1]$; see Figure 4.

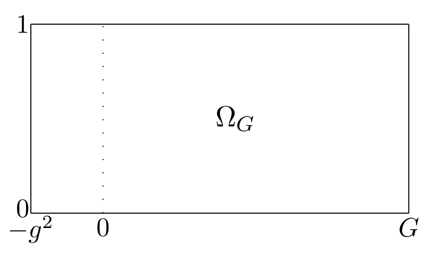

FIGURE 4. The domain Ω_G.

Proof. We will frequently use singularizations and insertions which are slightly different from the singularizations and insertions defined in the previous subsection:

Singularization. Let $A, B \in \mathbb{Z}$, $A \geq 0$, $B \geq 1$ and $\xi > -1$, then:

$$A + \cfrac{1}{1 + \cfrac{-1}{B+\xi}} = A + 1 + \frac{1}{B - 1 + \xi}.$$

Insertion. Let $A, B \in \mathbb{Z}$, $A \geq 0$, $B \geq 3$ and $\xi > -1$, then:

$$A + \frac{1}{B+\xi} = A + 1 + \cfrac{-1}{1 + \cfrac{1}{B - 1 + \xi}}.$$

Now let $1 < \alpha < G$, then we want to find the planar natural extension Ω_α. Let $x \in I_G$ and let $n \geq 0$ be the first "time" we have that $T_G^n(x) \geq \alpha$. So if we again define for $k \geq 0$: $(t_k, v_k) = \mathcal{T}_G^k(x, 0)$, then n is the first non-negative integer for which $(t_n, v_n) \in D_\alpha = [\alpha, G) \times [0, 1]$. Since $T_G(1) = 0$, we find that $\varepsilon_{n+2} = \text{sign}(T_G^{n+1}(x)) = -1$ and that a_{n+2} is at least equal to 3 (viz. $T_G([1, G]) = [-g^2, 0]$ and $1/g^2 = G^2 = G + 1$). But then the G-expansion of x is given by

$$x = [0; \varepsilon_1/a_1, \ldots, \varepsilon_n/a_n, 1/1, -1/a_{n+2}, \varepsilon_{n+3}/a_{n+3}, \ldots]. \tag{22}$$

As in the previous subsection we want to "skip" D_α; t_n is "too big". To achieve this, we singularize $a_{n+1} = 1$, to find the following continued fraction expansion of x:

$$x = [0; \varepsilon_1/a_1, \ldots, \varepsilon_n/a_n + 1, 1/a_{n+2} - 1, \varepsilon_{n+3}/a_{n+3}, \ldots]. \tag{23}$$

Now both $a_n + 1$ and $a_{n+2} - 1$ are even, so (23) is **not** an odd continued fraction expansion of x. We need an insertion of $-1/1$ to arrive at

$$x = [0; \varepsilon_1/a_1, \ldots, \varepsilon_n/a_n + 2, -1/1, 1/a_{n+2} - 2, \varepsilon_{n+3}/a_{n+3}, \ldots]. \tag{24}$$

If we denote this last expansion (24) of x as $x = [a_0^*; \varepsilon_1^*/a_1^*, \ldots]$, then we have that $a_0^* = 1$ if $n = 0$, and $a_0^* = 0$ otherwise, and that $\varepsilon_1^* = \varepsilon_1, \ldots, \varepsilon_n^* = \varepsilon_n$, $\varepsilon_{n+1}^* = -1$, $\varepsilon_{n+2}^* = 1$, and that $\varepsilon_{n+k}^* = \varepsilon_{n+k}$, for $k \geq 3$. Similarly, for the partial quotients we have that $a_1^* = a_1, \ldots, a_{n-1}^* = a_{n-1}$, $a_n^* = a_n + 2$, $a_{n+1}^* = 1$, $a_{n+2}^* = a_{n+2} - 2$, and that $a_{n+k}^* = a_{n+k}$, for $k \geq 3$. As in the previous subsection we can find the relation between (t_n, v_n) and (t_n^*, v_n^*), and (t_{n+1}, v_{n+1}) and (t_{n+1}^*, v_{n+1}^*); we skip the details, as these are similar to what we did in the previous subsection, and essentially applying the above steps of first a singularization and then a singularization to t_n and v_n. As in the previous subsection we have that

$$(t_n^*, v_n^*) = \left(t_n - 2, \frac{v_n}{1 + 2v_n}\right).$$

In view of this we define for $1 < \alpha < G$ a map $M_\alpha : \Omega_G \to \mathbb{R}^2$ similar to the definition of M_α from (19):

$$M_\alpha(t, v) = \begin{cases} (t, v), & \text{if } (t, v) \in \Omega_G \setminus D_\alpha, \\ \\ \left(t - 2, \dfrac{v}{1 + 2v}\right), & \text{if } (t, v) \in D_\alpha. \end{cases} \tag{25}$$

Since $G - 2 = -g^2$, it follows that $M_\alpha(D_\alpha) = [\alpha - 2, -g^2) \times [0, \frac{1}{3}]$; see also Figure 5.

As in the previous subsection we find that

$$(t_{n+1}^*, v_{n+1}^*) = \mathcal{T}_\alpha(t_n^*, v_n^*),$$

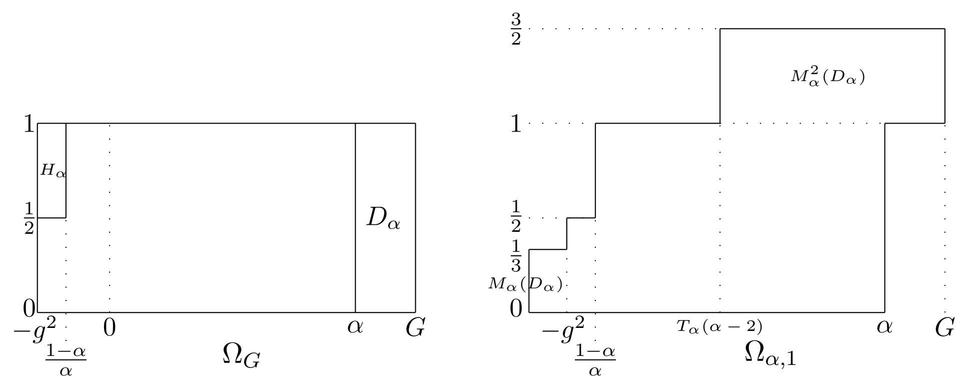

FIGURE 5. The domain Ω_G and the region $\Omega_{\alpha,1}$.

and therefore we extend the definition of M_α to $M_\alpha(D_\alpha)$ by

$$M_\alpha(t,v) = \left(\frac{-1}{t} - 1, \frac{1}{1-v}\right) (= \mathcal{T}_\alpha((t,v))),$$

Note that M_α preserves the invariant measure with density $(3\log G)^{-1}(1+xy)^{-2}$ both on D_α and on $M_\alpha(D_\alpha)$. We have that

$$M_\alpha^2(D_\alpha) = [T_\alpha(\alpha-2), G) \times [1, \tfrac{3}{2}].$$

Removing D_α and $H_\alpha = \mathcal{T}_G(D_\alpha) = [-g^2, \frac{1-\alpha}{\alpha}) \times [\frac{1}{2}, 1]$ from Ω_G we find as a new planar domain,

$$\Omega_{\alpha,1} = (\Omega_G \setminus (D_\alpha \cup H_\alpha)) \cup M_\alpha(D_\alpha) \cup M_\alpha^2(D_\alpha).$$

One might be inclined to think that $\Omega_{\alpha,1}$ is the planar natural extension of T_α but this is not correct. Note that $\Omega_{\alpha,1}$ contains the rectangle $D_{\alpha,1} = [\alpha, G) \times [1, \frac{3}{2})$. Extending the definition of M_α in an obvious way to $D_{\alpha,1}$ and $M_\alpha(D_{\alpha,1})$ we see that

$$M_\alpha(D_{\alpha,1}) = [\alpha-2, -g^2) \times [\tfrac{1}{3}, \tfrac{3}{8}] \quad \text{and} \quad M_\alpha^2(D_{\alpha,1}) = [T_\alpha(\alpha-2), G) \times [\tfrac{3}{2}, \tfrac{8}{5}];$$

Removing $D_{\alpha,1}$ and $H_{\alpha,2} = \mathcal{T}_G(D_{\alpha,1}) = [-g^2, \frac{1-\alpha}{\alpha}) \times [\frac{2}{5}, \frac{1}{2}]$ from $\Omega_{\alpha,1}$ we find as a new planar domain

$$\Omega_{\alpha,2} = (\Omega_{\alpha,1} \setminus (D_{\alpha,1} \cup H_{\alpha,1})) \cup M_\alpha(D_{\alpha,1}) \cup M_\alpha^2(D_{\alpha,1}),$$

see Figure 6.

We find that $\Omega_{\alpha,2}$ is also not the planar natural extension of T_α; the domain $\Omega_{\alpha,2}$ contains the rectangle $D_{\alpha,2} = [\alpha, G) \times [\frac{3}{2}, \frac{8}{5})$, which has a first coordinate which is "too large". If we now repeat the above procedure we find a sequence of rectangles $D_{\alpha,2}, D_{\alpha,3}, \ldots, D_{\alpha,n}, \ldots$ such that if $D_{\alpha,0} = D_\alpha = [\alpha, G) \times [0,1]$, and for $n \geq 0$,

$$D_{\alpha,n} = [\alpha, G) \times [c_{2n-2}, c_{2n}],$$

where $(c_n)_{n\geq 0}$ is the sequence of regular continued fraction expansion convergents of G.

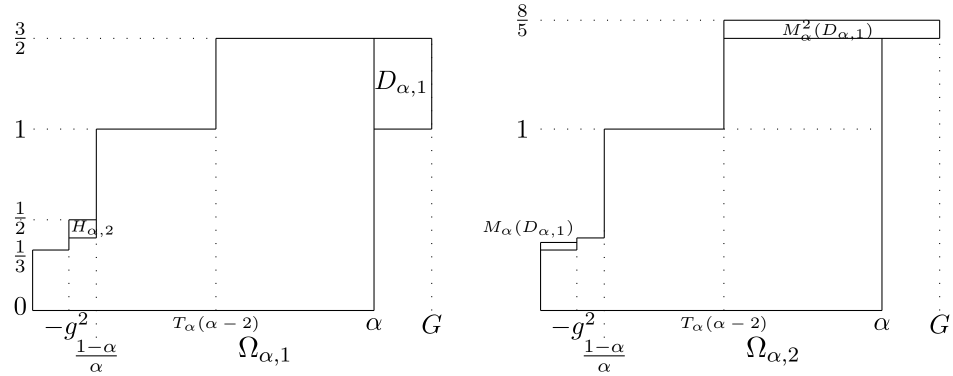

FIGURE 6. The regions $\Omega_{\alpha,1}$ and $\Omega_{\alpha,2}$.

To see why this is indeed the case, we define the map $M_\alpha : [\alpha, G) \times [0, G] \cup [\alpha - 2, -g^2) \times [0, g) \to \mathbb{R}^2$ as

$$M_\alpha(t,v) = \begin{cases} \left(t-2, \frac{v}{1+2v}\right), & \text{if } (t,v) \in [\alpha, G) \times [0, G]; \\ \\ \left(\frac{-1}{t} - 1, \frac{1}{1-v}\right), & \text{if } (t,v) \in [\alpha-2, -g^2) \times [0, g]. \end{cases}$$

So for $t = G$ and $0 \le v \le G$ we find that

$$M_\alpha^2(t,v) = (G, m(v))\,,$$

where the second coordinate map $m(v)$ satisfies $m(0) = 1$ and for $0 < v \le G$:

$$m(v) = \frac{1}{1 - \dfrac{v}{1+2v}} = \frac{2v+1}{v+1} = 1 + \frac{v}{1+v} = 1 + \cfrac{1}{1 + \cfrac{1}{v}}.$$

But then we have for $0 < v \le G$:

$$m^2(v) = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{v}}}}.$$

Setting $m_0 = 0$, $m_1 = m(0) = 1$, $m_2 = m^2(0) = m(1) = \frac{3}{2}$, $\dots$, $m_n = m^n(0) = m^{n-1}(1)$, for $n \ge 1$, we see that

$$m_n = [1;\, \underbrace{1, 1, \dots, 1}_{2(n-1) \text{ times}}\,] \uparrow G, \quad \text{if } n \to \infty. \tag{26}$$

We denote the regular continued fraction convergent of G in (26) by $[1; 1^{2n-2}]$. Setting

$$D_{\alpha,n} = [\alpha, G) \times [m_n, m_{n+1}], \quad \text{for } n \ge 0$$

then by construction we have that

$$M_\alpha^2(D_{\alpha,n}) = [T_\alpha(\alpha-2), G) \times [m_{n+1}, m_{n+2}], \quad \text{for } n \ge 0,$$

which contains $D_{\alpha,n+1}$ since $T_\alpha(\alpha-2)\le\alpha$. Furthermore, we see that at each time-step n we add as a "new" region:

$$M_\alpha(D_{\alpha,n}) = [\alpha-2,-g^2)\times\left[\frac{m_n}{1+2m_n},\frac{m_{n+1}}{1+2m_{n+1}}\right].$$

Note that for $0\le v\le G$ we have that:

$$\frac{v}{1+2v}=\frac{1}{2+\dfrac{1}{v}},$$

and therefore for $n\ge 1$ we have that

$$\frac{m_n}{1+2m_n}=[0;2,1^{2n-1}]\uparrow g^2,\quad \text{as } n\to\infty.$$

Note that the natural extension map $\mathcal{T}_G$ is not defined on $\bigcup_{n=1}^{\infty} D_{\alpha,n}=[\alpha,G)\times(1,G]$. We extend the definition of $\mathcal{T}_G$ on this set in the obvious way:

$$\mathcal{T}_G(t,v)=\left(\frac{1}{t}-1,\frac{1}{1+v}\right),\quad \text{for } (t,v)\in[\alpha,G)\times(1,G].$$

At every "time" n in our construction we add $M_\alpha(D_{\alpha,n})$ and $M_\alpha^2(D_{\alpha,n})$ to $\Omega_{\alpha,n}$, and remove from $\Omega_{\alpha,n}$ the rectangles $D_{\alpha,n}$ and $\mathcal{T}_G(D_{\alpha,n})$. Note that this last set is equal to:

$$\mathcal{T}_G(D_{\alpha,n})=\Big(-g^2,\frac{1}{\alpha}-1\Big]\times\left[\frac{1}{1+m_{n+1}},\frac{1}{1+m_n}\right],\qquad \text{for } n\ge 0.$$

Note that the sequence $(\frac{1}{1+m_n})_{n\ge 0}$ is a monotonically decreasing and bounded sequence from below by g^2, and that we have that

$$\lim_{n\to\infty}\frac{1}{1+m_n}=\frac{1}{1+G}=g^2.$$

Therefore the set

$$\bigcup_{n=0}^{\infty}\mathcal{T}_G(D_{\alpha,n})=\mathcal{T}_G([\alpha,G)\times[0,G])=(-g^2,\tfrac{1}{\alpha}-1]\times[g^2,1];$$

will be removed from Ω_G in the limit as $n\to\infty$.

So repeating for $n\ge 0$ the process of adding $M_\alpha(D_{\alpha,n})$ and $M_\alpha^2(D_{\alpha,n})$ and deleting $D_{\alpha,n}$ and $\mathcal{T}_G(D_{\alpha,n})$, we find in the limit as $n\to\infty$ that:

$$\Omega_\alpha=[\alpha-2,\frac{1}{\alpha}-1)\times[0,g^2]\cup[\frac{1}{\alpha}-1,T_\alpha(\alpha-2))\times[0,1]\cup[T_\alpha(\alpha-2),\alpha)\times[0,G];$$

see Figure 7. This is exactly the domain of the natural extension of the map T_α for $1<\alpha<G$, as found by Boca and the fourth author in [1].

□

Again, note that for $\alpha\in(1,G)$ the map T_α is not expansive on part of its domain. However, due to the infinite sequence of isomorphisms we constructed for the natural extension for T_α, we have that T_α is still ergodic, showing that T_α is *eventually expanding* and ergodic.

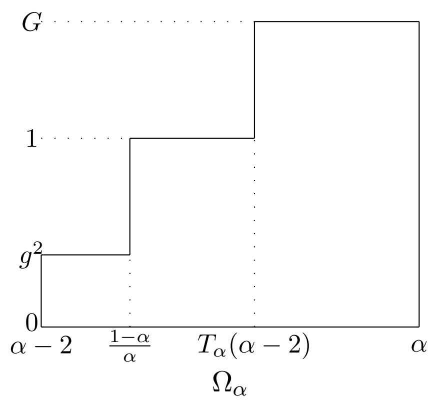

FIGURE 7. The domain Ω_α for $\alpha \in (1, G)$.

3. **The case** $\alpha \in [\frac{\sqrt{13}-1}{6}, g)$**.** In [1], Boca and the fourth author also found by simulation the natural extension for $\alpha = 0.9g$, which is a value of α outside their domain of possible α. In this section we will use the approach from the previous section to exactly determine the domain of natural extension of α for $\alpha \in [\frac{\sqrt{13}-1}{6}, g)$. We will see that these natural extensions are quite more complicated than those for $\alpha \in [g, G]$, which are essentially the unions of 2 or 3 rectangles in $\mathbb{R}^2$. Again, as in the previous section, the natural extensions we find will be isomorphic the natural extension due to Schweiger and Rieger (so the case $\alpha = 1$).

Theorem 3.1. *Let $\alpha \in [\frac{\sqrt{13}-1}{6}, g)$ and $x \in [\alpha-2, \alpha)$. The α-continued fraction expansion with odd partial quotients of x can be found by using singularizations and insertions. Via this method we also obtain the domain of the natural extension explicitly. Furthermore, we find that the dynamical system is metrically isomorphic to the system for $\alpha = g$.*

Proof. This time we start from Ω_g. For $\alpha = g$, Boca and the fourth author found in [1]

$$\Omega_g = \big([g-2, g) \times [0, g^2]\big) \cup \left([\frac{-g^2}{1+g^2}, g) \times [1, G]\right)$$

as planar domain for the natural extension $(\Omega_g, \bar{\mathcal{B}}_g, \bar{\mu}_g, \mathcal{T}_g)$; see Figure 8.

Now let $\alpha \in [\frac{1}{2}, g)$. Since $[\alpha, g) \not\subset I_\alpha$, we must consider those $(t_n, v_n) \in \Omega_g$ for which $t_n \geq \alpha$. So if $x \in I_g$ has g-expansion $x = [0; \varepsilon_1/a_1, \dots]$ and $n \geq 0$ is the first index for which $t_n \geq \alpha$, then $\varepsilon_{n+1} = +1$, $a_{n+1} = 3$, $\varepsilon_{n+2} = -1 = \varepsilon_{n+3}$, $a_{n+2} = 1$, and $a_{n+3} \geq 5$:

$$x = \cfrac{\varepsilon_1}{a_1 + \ddots + \cfrac{\varepsilon_n}{a_n + \cfrac{1}{3 + \cfrac{-1}{1 + \cfrac{-1}{a_{n+3} + \ddots}}}}}, \tag{27}$$

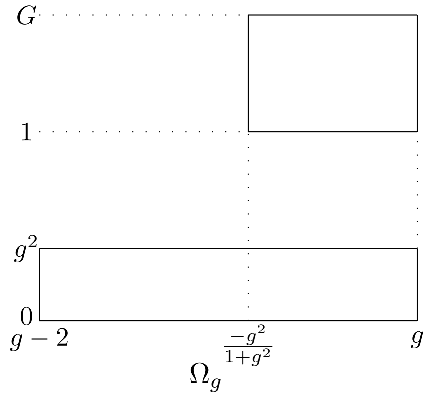

FIGURE 8. The domain Ω_g.

Removing the $-1/1$ directly after $a_{n+1}(=3)$ in (27) yields as an "intermediate" expansion of x:

$$x = \cfrac{\varepsilon_1}{a_1 + \ddots + \cfrac{\varepsilon_n}{a_n + \cfrac{1}{2 + \cfrac{-1}{a_{n+3} - 1 + \ddots}}}}. \tag{28}$$

Note that in (28) we have that $a_{n+1} - 1 = 2$, and that $a_{n+3} - 1$ is even, so (28) is *not* a continued fraction expansion of x with only odd partial quotients. Inserting $1/1$ "between" $a_{n+1} - 1(= 2)$ and $\frac{-1}{a_{n+3}-1+\dots}$ (in fact, this is undoing an earlier singularization of $1/1$) yields that

$$x = \cfrac{\varepsilon_1}{a_1 + \ddots + \cfrac{\varepsilon_n}{a_n + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{a_{n+3} - 2 + \ddots}}}}}. \tag{29}$$

Now (29) is a continued fraction expansion of x with only odd partial quotients. However, it is **not** the α-expansion of x, since for these values of α we cannot have that a partial quotient is equal to 1 while its corresponding $\varepsilon_i = +1$. Singularizing now the first partial quotient equal to 1 after a_n yields:

$$x = \cfrac{\varepsilon_1}{a_1 + \ddots + \cfrac{\varepsilon_n}{a_n + 1 + \cfrac{-1}{2 + \cfrac{1}{a_{n+3} - 2 + \ddots}}}}. \tag{30}$$

We find that (30) is a continued fraction expansion of x of which some of the partial quotients are even. Inserting $-1/1$ after a_n+1 now yields:

$$x = \cfrac{\varepsilon_1}{a_1 + \ddots + \cfrac{\varepsilon_n}{a_n + 2 + \cfrac{-1}{1 + \cfrac{-1}{3 + \cfrac{1}{a_{n+3} - 2 + \ddots}}}}}. \tag{31}$$

If we denote the expansion (31) of x as $[a_0^*; \varepsilon_1^*/a_1^*, \varepsilon_2^*/a_2^*, \dots]$, we find that

$$t_n^* = [0; -1/1, -1/3, 1/a_{n+3} - 2, \dots] \tag{32}$$

and

$$v_n^* = [0; 1/a_n + 2, \varepsilon_n/a_{n-1}, \dots, \varepsilon_2/a_1]. \tag{33}$$

Recall that for the g-expansion of x we have that

$$t_n = [0; 1/3, -1/1, -1/a_{n+3}, \dots]$$

and

$$v_n = [0; 1/a_n, \varepsilon_n/a_{n-1}, \dots, \varepsilon_2/a_1].$$

But then we have, that:

$$\begin{aligned} t_n &= \cfrac{1}{3 + \cfrac{-1}{1 + \cfrac{-1}{a_{n+3} + \ddots}}} = \cfrac{1}{2 + \cfrac{-1}{a_{n+3} - 1 + \ddots}} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{a_{n+3} - 2 + \ddots}}} \\ &= 1 + \cfrac{-1}{2 + \cfrac{1}{a_{n+3} - 2 + \ddots}} = 1 + 1 + \cfrac{-1}{1 + \cfrac{-1}{3 + \cfrac{1}{a_{n+3} - 2 + \ddots}}} = 2 + t_n^*, \end{aligned}$$

from which we see that $t_n^* = t_n - 2$ (as was to be expected).

As in the previous sections we also have, since $v_{n-1} = q_{n-2}/q_{n-1}$, that

$$v_n^* = \frac{v_n}{1 + 2v_n},$$

and we see that

$$(t_n^*, v_n^*) = \left(t_n - 2, \frac{v_n}{1 + 2v_n}\right),$$

and in view of this we define sets[4] $D_{\alpha,\ell} = [\alpha, g) \times [0, g^2]$, $D_{\alpha,u} = [\alpha, g) \times [1, G]$, and a map $M_\alpha : \Omega_g \to \mathbb{R}^2$ by

$$M_\alpha(t, v) = \begin{cases} (t, v), & \text{if } (t, v) \notin D_{\alpha,\ell} \cup D_{\alpha,u}; \\ \left(t - 2, \frac{v}{2v+1}\right), & \text{if } (t, v) \in D_{\alpha,\ell} \cup D_{\alpha,u}; \end{cases}$$

see Figure 9. We now "delete" $D_{\alpha,\ell}$ and $D_{\alpha,u}$ from Ω_g, and add the new regions $M_\alpha(D_{\alpha,\ell})$ and $M_\alpha(D_{\alpha,u})$ to Ω_g. These regions are:

$$M_\alpha(D_{\alpha,\ell}) = [\alpha - 2, g - 2] \times \left[0, \frac{1}{3 + G}\right], \text{ and } \quad M_\alpha(D_{\alpha,u}) = [\alpha - 2, g - 2] \times \left[\frac{1}{3}, g^2\right].$$

[4] l for lower and u for upper.

Note that by "removing" $D_{\alpha,\ell}$ and $D_{\alpha,u}$ we create "holes" in the lower rectangle in Ω_g:

$$\mathcal{T}_g(D_{\alpha,\ell}) = \left[g-2, \frac{1-3\alpha}{\alpha}\right] \times \left[\frac{1}{4-g}, \frac{1}{3}\right]$$

and

$$\mathcal{T}_g(D_{\alpha,u}) = \left[g-2, \frac{1-3\alpha}{\alpha}\right] \times \left[\frac{1}{3+G}, \frac{1}{4}\right].$$

In their turn, these hole create holes in the upper rectangle of Ω_g:

$$\mathcal{T}_g^2(D_{\alpha,\ell}) = \left[\frac{-g^2}{1+g^2}, \frac{1-2\alpha}{3\alpha-1}\right] \times \left[\frac{4-g}{3-g}, \frac{3}{2}\right],$$

and

$$\mathcal{T}_g^2(D_{\alpha,u}) = \left[\frac{-g^2}{1+g^2}, \frac{1-2\alpha}{3\alpha-1}\right] \times \left[\frac{3+G}{2+G}, \frac{4}{3}\right].$$

Note that for $\alpha \in [\frac{\sqrt{13}-1}{6}, g)$ we have that

$$\frac{-g^2}{1+g^2} < \frac{1-2\alpha}{3\alpha-1} \leq \alpha,$$

and that $\frac{1-2\alpha}{3\alpha-1} = \alpha$ yields that $\alpha = \frac{\sqrt{13}-1}{6} = 0.434258545\ldots$; for this value of α we have that the "upper rectangle" has two "holes" running all the way through the rectangle.

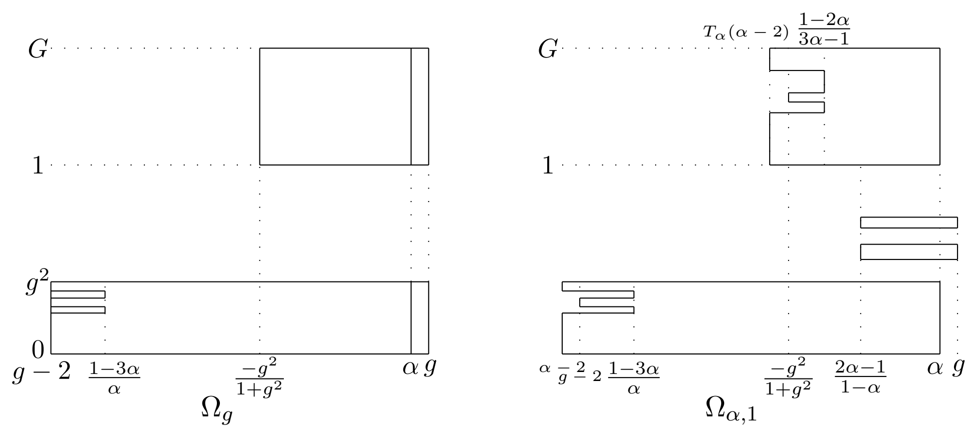

FIGURE 9. The domain Ω_g and the region $\Omega_{\alpha,1}$.

From (32) it follows that

$$\mathcal{T}_\alpha(t_n^*) = \frac{-1}{t_n^*} - 1 = [0; -1/3, 1/a_{n+2} - 2, \ldots] = t_{n+1}^*, \tag{34}$$

and from (33) we see that

$$v_{n+1}^* = [0; 1/1, -1/a_n + 2, \varepsilon_n/a_{n-1}, \ldots, \varepsilon_2/a_1] = \frac{1}{1-v_n^*}; \tag{35}$$

so it follows that

$$(t_{n+1}^*, v_{n+1}^*) = \mathcal{T}_\alpha(t_n^*, v_n^*).$$

We have that two rectangles with "new" areas are attached to the "upper rectangle" in Ω_g:

$$N_{\alpha,\ell} := \mathcal{T}_\alpha(M_\alpha(D_{\alpha,\ell})) = \left[T_\alpha(\alpha-2), \frac{-g^2}{1+g^2}\right] \times \left[1, \frac{3+G}{2+G}\right]$$

and

$$N_{\alpha,u} := \mathcal{T}_\alpha(M_\alpha(D_{\alpha,u})) = \left[T_\alpha(\alpha-2), \frac{-g^2}{1+g^2}\right] \times \left[\frac{3}{2}, G\right].$$

In view of (34) and (35) we now apply the auxiliary map

$$\mathcal{A}(\xi,\eta) = \left(-\frac{1}{\xi} - 3, \frac{1}{3-\eta}\right)$$

to $N_{\alpha,\ell}$ and $N_{\alpha,u}$ (note that on these two regions this map is actually $\mathcal{T}_g$), yielding two disjoint "islands" (and here we used that $T_\alpha(\alpha-2) = \frac{\alpha-1}{2-\alpha}$):

$$\mathcal{A}(N_{\alpha,\ell}) = \left[\frac{2\alpha-1}{1-\alpha}, g\right] \times \left[\frac{1}{2}, \frac{2+G}{3+2G}\right]$$

and

$$\mathcal{A}(N_{\alpha,u}) = \left[\frac{2\alpha-1}{1-\alpha}, g\right] \times \left[\frac{2}{3}, \frac{1}{3-G}\right];$$

see Figure 9. Note in particular how well the "new part" and the "holes" align.

As in Section 2.3, the domain $\Omega_{\alpha,1}$, given by

$$\begin{aligned}\Omega_{\alpha,1} \quad = \quad & \Omega_g \setminus (\mathcal{T}_g(D_{\alpha,\ell} \cup D_{\alpha,u}) \cup \mathcal{T}_g^2(D_{\alpha,\ell} \cup D_{\alpha,u})) \\ & \cup M_\alpha(D_{\alpha,\ell} \cup D_{\alpha,u}) \cup \mathcal{T}_\alpha(M_\alpha(D_{\alpha,\ell} \cup D_{\alpha,u})) \\ & \cup \mathcal{A}(\mathcal{T}_\alpha(M_\alpha(D_{\alpha,\ell} \cup D_{\alpha,u})))\end{aligned}$$

is **not** a version of the natural extension for α; the reason is, that since $T_\alpha(\alpha-2) \le \alpha$ the two rectangles $\mathcal{A}(N_{\alpha,\ell})$ and $\mathcal{A}(N_{\alpha,u})$ have values on their first coordinates which are larger than α; see also Figure 9. In view of this we define new regions $D_{\alpha,\ell,1}$ and $D_{\alpha,u,1}$ which must both be moved and deleted in a way similar to what we just did:

$$D_{\alpha,\ell,1} = [\alpha, g) \times \left[\frac{1}{2}, \frac{2+G}{3+2G}\right], \text{ and } D_{\alpha,u,1} = [\alpha, g) \times \left[\frac{2}{3}, \frac{1}{3-G}\right].$$

So as in Section 2.3 we need to repeat (infinitely often) the above procedure, but now to $D_{\alpha,\ell,1}$ and $D_{\alpha,u,1}$. We will describe one more "round," and then give the general situation, which can be proved by induction. Extending the domains of the maps we defined in this section (M_α and $\mathcal{A}$) in the obvious way, we find that two new added regions are:

$$M_\alpha(D_{\alpha,\ell,1}) = [\alpha-2, g-2) \times \left[\frac{1}{4}, \frac{G+2}{4G+7}\right]$$

and

$$M_\alpha(D_{\alpha,u,1}) = [\alpha-2, g-2) \times \left[\frac{2}{7}, \frac{1}{4-g}\right]$$

(note that these new rectangles align well with the existing "holes"). By "removing" $D_{\alpha,\ell,1}$ and $D_{\alpha,u,1}$ we create new "holes" in the lower rectangle in Ω_g:

$$\mathcal{T}_g(D_{\alpha,\ell,1}) = \left[g-2, \frac{1-3\alpha}{\alpha}\right] \times \left[\frac{2G+2}{7G+11}, \frac{2}{7}\right]$$

and

$$\mathcal{T}_g(D_{\alpha,u,1}) = \left[g-2, \frac{1-3\alpha}{\alpha}\right] \times \left[\frac{2-g}{7-3g}, \frac{3}{11}\right].$$

Since $\frac{2-g}{7-3g} = 0.26855\cdots = \frac{G+2}{4G+7}$ we see again an alignment between new rectangles and new holes. In their turn, these new holes create holes in the upper rectangle of Ω_g:

$$\mathcal{T}_g^2(D_{\alpha,\ell,1}) = \left[\frac{-g^2}{1+g^2}, \frac{1-2\alpha}{3\alpha-1}\right] \times \left[\frac{7G+11}{5G+8}, \frac{7}{5}\right],$$

and

$$\mathcal{T}_g^2(D_{\alpha,u,1}) = \left[\frac{-g^2}{1+g^2}, \frac{1-2\alpha}{3\alpha-1}\right] \times \left[\frac{4G+7}{3G+5}, \frac{11}{8}\right].$$

Again we have that two rectangles with "new" areas are attached to the 'upper rectangle' in Ω_g:

$$N_{\alpha,\ell,1} := \mathcal{T}_\alpha(M_\alpha(D_{\alpha,\ell})) = \left[T_\alpha(\alpha-2), \frac{-g^2}{1+g^2}\right] \times \left[\frac{2G+1}{2G}, \frac{4}{3}\right]$$

and

$$N_{\alpha,u,1} := \mathcal{T}_\alpha(M_\alpha(D_{\alpha,u})) = \left[T_\alpha(\alpha-2), \frac{-g^2}{1+g^2}\right] \times \left[\frac{7}{5}, \frac{4-g}{3-g}\right].$$

Applying the auxiliary map $\mathcal{A}$ to these new rectangles $N_{\alpha,\ell,1}$ and $N_{\alpha,u,1}$ yields two disjoint "islands" located in between the previous two "islands":

$$\mathcal{A}(N_{\alpha,\ell,1}) = \left[\frac{2\alpha-1}{1-\alpha}, g\right] \times \left[\frac{2}{4-g}, \frac{3}{5}\right]$$

and

$$\mathcal{A}(N_{\alpha,u,1}) = \left[\frac{2\alpha-1}{1-\alpha}, g\right] \times \left[\frac{3-g}{5-2g}, \frac{5}{8}\right].$$

As for $\Omega_{\alpha,1}$, the domain $\Omega_{\alpha,2}$, given by

$$\begin{aligned}\Omega_{\alpha,2} \quad = \quad & \Omega_{\alpha,1} \setminus (\mathcal{T}_g(D_{\alpha,\ell,1} \cup D_{\alpha,u,1}) \cup \mathcal{T}_g^2(D_{\alpha,\ell,1} \cup D_{\alpha,u,1})) \\ & \cup M_\alpha(D_{\alpha,\ell,1} \cup D_{\alpha,u,1}) \cup \mathcal{T}_\alpha(M_\alpha(D_{\alpha,\ell,1} \cup D_{\alpha,u,1})) \\ & \cup \mathcal{A}(\mathcal{T}_\alpha(M_\alpha(D_{\alpha,\ell,1} \cup D_{\alpha,u,1})))\end{aligned}$$

is **not** a version of the natural extension for α; again two rectangles are "sticking out":

$$D_{\alpha,\ell,2} = [\alpha, g] \times \left[\frac{2}{4-g}, \frac{3}{5}\right] \text{ and } D_{\alpha,u,2} = [\alpha, g] \times \left[\frac{3-g}{5-2g}, \frac{5}{8}\right].$$

By repeating the above procedure, we define for $n \geq 1$ nested disjoint rectangles $D_{\alpha,\ell,n}$ and $D_{\alpha,u,n}$.

Note, that for $(x,y) \in D_{\alpha,\ell} \cup D_{\alpha,u} \cup D_{\alpha,\ell,1} \cup D_{\alpha,u,1}$ we used various maps to the second coordinate y:

$$y \mapsto \frac{y}{2y+1} \mapsto \frac{1}{1-\frac{y}{2y+1}} = \frac{2y+1}{y+1} \mapsto \frac{1}{3-\frac{2y+1}{y+1}} = \frac{y+1}{y+2} = \cfrac{1}{1+\cfrac{1}{1+y}}, \tag{36}$$

under the map $\mathcal{T}_\alpha$, and we see that the function $V : [0, G] \to \mathbb{R}$, defined by

$$V(x) = \frac{1+x}{2+x},$$

is monotonically increasing, with fixed point g. Setting $D_{\alpha,\ell,0} := D_{\alpha,\ell}$ and $D_{\alpha,u,0} := D_{\alpha,u}$, we see for $n \geq 0$ that:

$$D_{\alpha,\ell,n} = \Big[\alpha, g\Big) \times V^n\left([0,g^2]\right), \text{ and } D_{\alpha,u,n} = \Big[\alpha, g\Big) \times V^n\left([1,G]\right).$$

Setting for $n \geq 0$:

$$V^n\left([0,g^2]\right) = [\ell_n, u_n], \quad \text{and} \quad V^n\left([1,G]\right) = [L_n, U_n],$$

so $\ell_0 = 0$, $u_0 = g^2$, $L_0 = 1$ and $U_0 = G$, we see that it follows from (36) that the regular continued fraction expansions of ℓ_n, u_n, L_n and U_n can be explicitly given; for $n \geq 1$:

$$\ell_n = [0; 1^{2n}] \;\; (= [0; \underbrace{1,1,\dots,1}_{2n-\text{times}}]), \quad u_n = [0; 1^{2n}, 2, \overline{1}\,], \quad L_n = [0; 1^{2n+1}],$$

(the bar indicates the periodic part; here we used that $g^2 = [0; 2, \overline{1}\,]$), and

$$U_n = [0; 1^{2n-1}, 2, \overline{1}\,]$$

(here we used that $G = [1; \overline{1}\,] = 1 + [0; \overline{1}\,]$). Elementary properties of regular continued fraction expansions now yield that for $n \geq 0$:

$$\ell_n < u_n < \ell_{n+1}, \quad L_{n+1} < U_{n+1} < L_n,$$

and

$$\ell_n \uparrow g, \quad \text{resp.} \quad L_n \downarrow g, \quad \text{as } n \to \infty.$$

Due to this, and due to the fact that the maps involved (M_α, $\mathcal{T}_g$, $\mathcal{T}_\alpha$ and $\mathcal{A}$) are all bijective a.s. we see that $(M_\alpha(D_{\alpha,\ell,n}))_n$ is a disjoint sequence of rectangles, converging from below to the line segment $[\alpha-2, g-2) \times \{\frac{1}{G+2}\}$, and that $(M_\alpha(D_{\alpha,u,n}))_n$ is a disjoint sequence of rectangles, also converging (but now from above) to the same line segment $[\alpha-2, g-2) \times \{\frac{1}{G+2}\}$. Due to this we have that $(\mathcal{T}_\alpha(M_\alpha(D_{\alpha,\ell,n})))_n$ is a disjoint sequence of rectangles, converging from below to the line segment $[T_\alpha(\alpha-2), T_g(g-2)) \times \{1+g^2\}$, and that $(T_\alpha(M_\alpha(D_{\alpha,u,n})))_n$ is a disjoint sequence of rectangles, also converging (but now from above) to the same line segment $[T_\alpha(\alpha-2), T_g(g-2)) \times \{1+g^2\}$.

For $n \geq 1$ we have that:

$$M_\alpha(D_{\alpha,\ell,n}) = [\alpha-2, g-2) \times \left[\frac{\ell_n}{2\ell_n+1}, \frac{u_n}{2u_n+1}\right],$$

where by definition of M_α we have that

$$\frac{\ell_n}{2\ell_n+1} = \cfrac{1}{2+\cfrac{1}{\ell_n}} = [0; 3, 1^{2n-1}].$$

In the same way we see that

$$\frac{u_n}{2u_n+1} = [0; 3, 1^{2n-1}, 2, \overline{1}\,], \quad \frac{L_n}{2L_n+1} = [0; 3, 1^{2n}], \quad \frac{U_n}{2U_n+1} = [0; 3, 1^{2n-2}, 2, \overline{1}\,].$$

Using (36) we can obtain in a similar way the regular continued fraction expansions of all endpoints of the second coordinate interval of all of the other added or deleted rectangles. For example, for $n \geq 1$ we have that

$$\mathcal{T}_g(D_{\alpha,\ell,n}) = \left(g-2, \frac{1-3\alpha}{\alpha}\right] \times \left[\frac{1}{3+u_n}, \frac{1}{3+\ell_n}\right]$$

and

$$\mathcal{T}_g(D_{\alpha,u,n}) = \Big(g-2, \frac{1-3\alpha}{\alpha}\Big] \times \Big[\frac{1}{3+U_n}, \frac{1}{3+L_n}\Big],$$

where

$$\frac{1}{3+u_n} = [0;3,1^{2n},2,\overline{1}], \quad \frac{1}{3+\ell_n} = [0;3,1^{2n}],$$

and

$$\frac{1}{3+U_n} = [0;3,1^{2n-1},2,\overline{1}], \quad \frac{1}{3+L_n} = [0;3,1^{2n+1}].$$

Recall we already saw that:

$$\mathcal{T}_g(D_{\alpha,\ell}) = \Big(g-2, \frac{1-3\alpha}{\alpha}\Big] \times \Big[\frac{1}{4-g}, \frac{1}{3}\Big]$$

and that

$$\mathcal{T}_g(D_{\alpha,u}) = \Big(g-2, \frac{1-3\alpha}{\alpha}\Big] \times \Big[\frac{1}{3+G}, \frac{1}{4}\Big].$$

Note that the added intervals and the deleted intervals "align" (see also Figure 10).

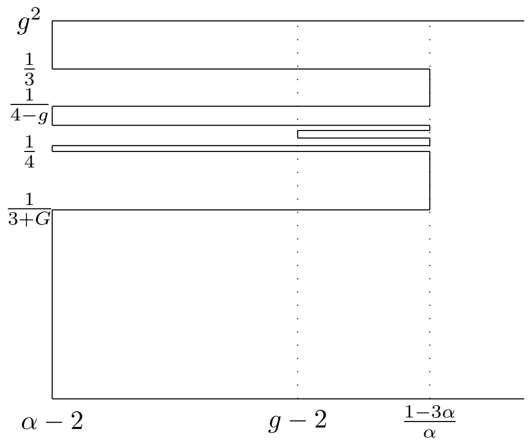

FIGURE 10. The left part of the natural extension after "one more round" for $\alpha \in [\frac{\sqrt{13}-1}{6}, g)$.

Setting $D_{\alpha,\ell,0} = D_{\alpha,\ell}$ and $D_{\alpha,u,0} = D_{\alpha,u}$, we have obtained for $\alpha \in [\frac{\sqrt{13}-1}{6}, g)$ following domain.

$$\begin{aligned}\Omega_\alpha \;=\;& \Big(\bigcup_{n=0}^{\infty} M_\alpha(D_{\alpha,\ell,n} \cup D_{\alpha,u,n})\Big) \cup \Big(\bigcup_{n=0}^{\infty} \mathcal{T}_\alpha(M_\alpha(D_{\alpha,\ell,n} \cup D_{\alpha,u,n}))\Big)\\
&\cup \Big([g-2,\alpha) \times [0,g^2]\Big) \setminus \Big(\bigcup_{n=0}^{\infty} \mathcal{T}_g(D_{\alpha,\ell,n} \cup D_{\alpha,u,n})\Big)\\
&\cup \Big([\frac{-1}{G+2},\alpha) \times [1,G]\Big) \setminus \Big(\bigcup_{n=0}^{\infty} \mathcal{T}_g^2(D_{\alpha,\ell,n} \cup D_{\alpha,u,n})\Big)\\
&\cup \Big(\bigcup_{n=0}^{\infty} \mathcal{A}\Big(\mathcal{T}_\alpha(M_\alpha(D_{\alpha,\ell,n} \cup D_{\alpha,u,n}))\Big)\Big).\end{aligned}$$

As we obtained our result for $[\frac{1}{2}, g)$, we are left to show it for $\alpha \in \left[\frac{\sqrt{13}-1}{6}, \frac{1}{2}\right)$. We repeat the procedure we have used now several times, starting with $\alpha = \frac{1}{2}$, and choosing $\frac{\sqrt{13}-1}{6} \le \alpha < \frac{1}{2}$. Since $T^2_{\frac{1}{2}}(\frac{1}{2}) = 0$, the situation is slightly different from the case where $\frac{1}{2} \le \alpha < g$: if $x \in I_{\frac{1}{2}}$ has $\frac{1}{2}$-expansion $x = [0; \varepsilon_1/a_1, \dots]$ and $n \ge 0$ is the first index for which $t_n \ge \alpha$, then $\varepsilon_{n+1} = +1$, $a_{n+1} = 3$, $\varepsilon_{n+2} = -1$, $a_{n+2} = 1$ and (and this is different from the case $\frac{1}{2} \le \alpha < g$): $\varepsilon_{n+3} = +1$, $a_{n+3} \ge 3$. What is also different from the case $\frac{1}{2} \le \alpha < g$, is that we only need to do the procedure once; now

$$D_\alpha = [\alpha, \tfrac{1}{2}) \times \bigcup_{n=0}^{\infty} \left([\ell_n, u_n] \cup [L_n, U_n]\right).$$

Details are left to the reader. □

Remark 3.2. Note that we must have in Theorem 1.1 that $\frac{\sqrt{13}-1}{6} \le \alpha < g$; in case $\alpha = \frac{\sqrt{13}-1}{6}$ one can see that $T^2_g(\alpha) = \alpha$, and that the "hole" in the top rectangle of Ω_g will be "all the way through." Clearly a new situation arises for values of α smaller than $\frac{\sqrt{13}-1}{6}$, which can be addressed by our approach, but things are becoming even more tedious for such values.

4. **Entropy.** Entropy as a function of α is widely studied for several different families of continued fractions. Nakada's α-continued fractions [4, 14, 16, 18, 20], Ito and Tanaka's α-continued fractions [5, 29], and Katok and Ugarcovici's α-continued fractions [12, 13] are the best known examples. Various tools work in a similar way for these families (see [17], Section 3 for a general set up). In this section, we will compare the entropy function of Odd α-continued fractions with the entropy function of Nakada's α-continued fractions (see Figure 11 for both functions). Nakada's α-continued fractions are the most studied. They were introduced in 1981 by Nakada [20] and brought back under attention by Luzzi and Marmi in 2008 (see [18]). For the Odd α-continued fractions the entropy is explicitly known on the interval $[\frac{\sqrt{13}-1}{6}, G]$ and equals $\frac{\pi^2}{9 \log G}$. This value is found in [1] for the interval $[g, G]$ and follows from the previous section for $\alpha \in (\frac{\sqrt{13}-1}{6}, g)$. For Nakada's α-continued fractions, the entropy is known on $[g^2, 1]$ and is given by $h(\alpha) = \frac{\pi^2}{6\log(1+\alpha)}$ for $\alpha \in (g, 1]$ and $h(\alpha) = \frac{\pi^2}{6\log(G)}$ for $\alpha \in [g^2, g]$; see [14, 16, 20]. Note that for both families, the entropy is not explicitly known on a large part of the parameter space, but obtained/estimated by simulation. We approximate the entropy by using Birkhoff averages $\frac{1}{n}\sum_{i=0}^{n-1} f \circ T^i_\alpha(x)$, where $f(x) = -2\log(|x|)$. We determine such average for many points and take the average of that. When the system is ergodic, these Birkhoff averages converge to $\int \log(|T'_\alpha(x)|)d\mu$. When the system satisfies Rényi's Condition this integral equals to the entropy of the dynamical system. This condition is often used to prove ergodicity.

In this article we proved ergodicity for $\alpha \in (\frac{\sqrt{13}-1}{6}, G]$. For the simulations of the entropy we asume ergodicity and Rényi's Condition hold for all values of α. We believe this is a reasonable asumption. We would like to add that the situation is more complicated than in the case of Nakada's α-expansions; for $\alpha \neq 1$ the map T_α is not expansive on part of its domain. We fixed this problem for $\alpha \in (\frac{\sqrt{13}-1}{6}, G]$ by constructing the natural extensions for the dynamical systems underlying these maps via isomorphisms essentially connecting them to the case $\alpha = 1$. We believe that this process, in which sometimes we use isomorphisms and (as in [11, 14]) induced transformations, can be extended to values of $\alpha \in (0, \frac{\sqrt{13}-1}{6})$. Another viable approach would be to show that the maps T_α are eventually expansive on their domain.

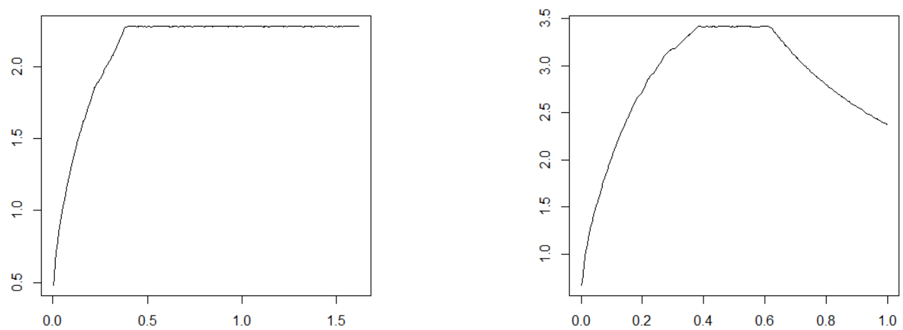

FIGURE 11. The entropy plotted as a function of α. On the left for the Odd α-continued fractions, on the right for Nakada's α-continued fractions. The values in these plots are obtained by simulations.

In [21], Nakada and Natsui proved that for Nakada's α-continued fractions there exists decreasing sequences of intervals $(I_n), (J_n), (K_n)$, (L_n) such that $\frac{1}{n} \in I_n$, $I_{n+1} < J_n < K_n < L_N < I_n$, and the entropy of $\hat{T}_\alpha$ is increasing on I_n, decreasing on K_n and constant on J_n and L_n. Here the map $\hat{T}_\alpha : [\alpha - 1, \alpha) \to [\alpha - 1, \alpha)$ is Nakada's α-continued fraction map defined as $\frac{1}{|x|} - \left\lfloor \frac{1}{|x|} + 1 - \alpha \right\rfloor$. Do we observe the same phenomena for the Odd α-continued fractions? In other words, can we also find decreasing sequences of intervals $(I_n), (J_n), (K_n)$, (L_n) with these properties? Figure 12 shows the entropy function of both the odd α-continued fractions and Nakada's α-continued fractions between $\frac{1}{4}$ and $\frac{1}{3}$.

For both types of continued fraction transformations, we observe intervals on which the entropy as a function of α is constant, as well as intervals where it is either increasing or decreasing. However the intervals on which the entropy is decreasing are more difficult to spot. A closer look around $\frac{14}{47}$ shows that there are also intervals on which the entropy of T_α is decreasing (see Figure 13).

The following result is analogous to the main result of [21] for Nakada's α-continued fractions:

Theorem 4.1. *There exists sequences of intervals* $(I_n), (J_n), (K_n), (L_n), n \geq 3$ *such that*

1. $\frac{1}{n} \in I_n$,
2. $I_{n+1} < J_n < K_n < L_n < I_n$,
3. *If there exists an ergodic, absolutely continuous,* T_α*-invariant measure, then the entropy of* T_α *is increasing on* I_n*, decreasing on* K_n *and constant on* J_n *and* L_n *for every* $n \geq 3$.

Here $I < J$ *means that any element of* J *is strictly larger than any element of* I.

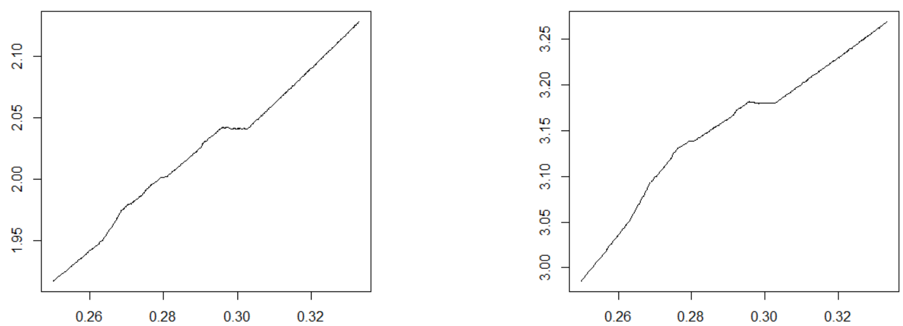

FIGURE 12. Simulations of the entropy as a function of α on the interval $[\frac{1}{4}, \frac{1}{3}]$. On the left for the odd α-continued fractions, on the right for Nakada's α-continued fractions.

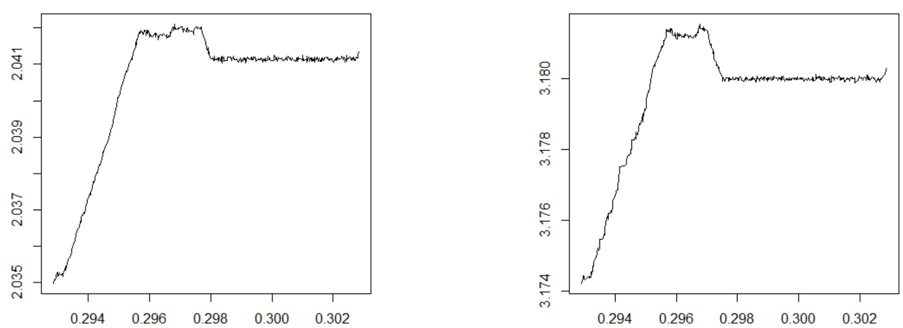

FIGURE 13. Simulations of the entropy as a function of α on an interval containing $\frac{14}{47}$. On the left for the odd α-continued fractions, on the right for Nakada's α-continued fractions.

We will prove this theorem using a series of lemmas with a prominent role reserved for a phenomena called *matching*. We say that matching holds for a parameter α if there are $N, M \in \mathbb{N}$ such that

$$T_\alpha^N(\alpha) = T_\alpha^M(\alpha - 2). \tag{37}$$

We call the minimal N and M for which (37) holds the *matching exponents*. The "matching property" is directly related with "moving pieces around" in the natural extension, as exhibited in Sections 2 and 3 of this paper. This process of relocating parts of the natural extension is also known as *quilting*, and has been studied by the second author and Tom Schmidt (and co-authors) in several papers; see e.g. [15]. The higher the matching exponents the more steps of quilting are needed.

For the families mentioned earlier, matching holds for almost every parameter. That is, for almost every $\alpha \in (0,1)$ there are $N, M \in \mathbb{N}$ such that $T_\alpha^N(\alpha) = T_\alpha^M(\alpha-1)$; see [4, 5, 6] and see also [3] where matching is studied for a different kind of continued fraction algorithm. Furthermore, the parameter space breaks up into intervals on which the matching exponents are constant. We can choose intervals $(I_n), (J_n), (K_n), (L_n)$ as such intervals (they will not cover the parameter space entirely). On such interval, if $N - M < 0$, the entropy of T_α is increasing; if $N - M = 0$, the entropy of T_α is constant; and if $N - M > 0$, the entropy of T_α is decreasing. It turns out that for our family of continued fractions we can also identify matching intervals and this property also holds. The proof is analogous to the other families (see [17]), and the connection between matching and entropy relies on the assumption of ergodicity.

Before we make sequences of intervals, we will first make sequences of rational numbers. For α equal to each of these rational numbers, we determine the matching exponents (N, M). Then we will show that for points in a small neighborhood of these rationals, we have matching exponents $(N+1, M+1)$. The sequences are chosen in a way such that the neighborhoods will give rise to the desired intervals $(I_n), (J_n), (K_n), (L_n)$.

Proposition 4.2. *Let $n \geq 3$ be an integer and define the following sequences:*

$$a_n = \frac{1}{n}, \quad b_n = \frac{2n+1}{2n^2+2n+1}, \quad c_n = \frac{5n-1}{n(5n-1)+5}, \quad d_n = \frac{n}{n^2+1}. \tag{38}$$

For these sequences we have $a_{n+1} < b_n < c_n < d_n < a_n$. Furthermore,

(i) for $\alpha = a_n$ we have that the matching exponents (N, M) are $(1, 3\frac{(n-1)}{2}+1)$ for n is odd and $(2, 3\frac{(n-2)}{2}+2)$ for n is even.

(ii) for $\alpha = b_n$ we have that the matching exponents (N, M) are $(3\frac{(n+1)}{2}+1, 3\frac{(n+1)}{2}+1)$ for n is odd and $(\frac{3n}{2}+1, \frac{3n}{2}+1)$ for n is even.

(iii) for $\alpha = c_n$ we have that the matching exponents (N, M) are $(3\frac{(n-1)}{2}+6, 3\frac{(n-1)}{2}+3)$ for n is odd and $(\frac{3n}{2}+4, \frac{3n}{2}+1)$ for n is even.

(iv) for $\alpha = d_n$ we have that the matching exponents (N, M) are $(3\frac{(n-1)}{2}+2, 3\frac{(n-1)}{2}+2)$ for n is odd and $(\frac{3n}{2}, \frac{3n}{2})$ for n is even.

Proof. We only will prove (i) for both n odd and even, and (ii) for n odd; all other cases have a similar proof. For $\alpha = a_n$, n odd we have $T_\alpha(\alpha) = 0$. For $\alpha - 2$ we find:

$$\begin{aligned} T_\alpha(\alpha-2) &= \frac{n}{2n-1} - 1 = \frac{1-n}{2n-1} \\ T_\alpha^2(\alpha-2) &= \frac{2n-1}{n-1} - 3 = \frac{2-n}{n-1} \\ T_\alpha^3(\alpha-2) &= \frac{n-1}{n-2} - 3 = \frac{5-2n}{n-2} = a_{n-2} - 2 \end{aligned}$$

(in case $n = 3$ we set $a_{n-2} = 1$). One can check that these three iterates of $\frac{1}{n} - 2$ are always in the interval $(\frac{1}{n} - 2, 0)$. Now

$$a_n - 2 < a_{n-2} - 2 < \cdots < a_3 - 2 < -1 < \frac{-n}{n+1}$$

so that for $k = 0, 1, \ldots, \frac{n-1}{2}$ the first digit of $a_{n-2k} - 2$ is -1. Since

$$T_\alpha(a_3 - 2) = \tfrac{3}{5} - 1 = -\tfrac{2}{5}, \quad T_\alpha^2(a_3-2) = \tfrac{5}{2} - 3 = -\tfrac{1}{2}, \quad T_\alpha^3(a_3-2) = 2 - 3 = -1$$

hold, we see that

$$T_\alpha^{3\frac{(n-1)}{2}}(\alpha-2)=-1, \quad \text{and so} \quad T_\alpha^{3\frac{(n-1)}{2}+1}(\alpha-2)=0.$$

In fact, the α-expansion of $\alpha-2$ for n odd is:

$$\alpha-2=[0;(-1,-3,-3)^{\frac{n-1}{2}},-1].$$

For $\alpha=a_n$, n even we have $T_\alpha(\alpha)=-1$ so $T_\alpha^2(\alpha)=0$. For $\alpha-2$, the first iterations will give the same digits as for n is odd, so we again find:

$$T_\alpha^3(\alpha-2)=a_{n-2}-2.$$

This gives

$$T_\alpha^{3\frac{(n-2)}{2}}(\alpha-2)=a_2-2=-\frac{3}{2} \quad \text{and so} \quad T_\alpha^{3\frac{(n-2)}{2}+2}(\alpha-2)=0.$$

Note that from this calculation, we also found the α-continued fraction expansion of $\alpha=a_n$ when n is even:

$$\alpha-2=[0;(-1,-3,-3)^{\frac{n-2}{2}},-1,-3].$$

This proves (i) both for n odd and even.

For $\alpha=b_n$, n odd we have

$$\begin{aligned}
T_\alpha(\alpha) &= \frac{2n^2+2n+1}{2n+1}-(n+2)=\frac{-(3n+1)}{2n+1}\\
T_\alpha^2(\alpha) &= \frac{2n+1}{3n+1}-1=\frac{-n}{3n+1}\\
T_\alpha^3(\alpha) &= \frac{3n+1}{n}-5=\frac{1}{n}-2=a_n-2.
\end{aligned}$$

In the last step we used that $b_n<a_n$. Again, by using $b_n<a_n$ and that $T_{a_n}^k(a_n-2)\leq 0$ for all $k\in\{1,\ldots,\frac{3(n-1)}{2}+1\}$ we find that for these values of k, $T_{a_n}^k(a_n-2)=T_{b_n}^k(a_n-2)$, thus

$$b_n=[0;(n+2),-1,-5,(-1,-3,-3)^{\frac{n-1}{2}},-1].$$

This yields that $T_\alpha^{\frac{3(n+1)}{2}+1}(\alpha)=0$.

For $\alpha-2$, write $n=2k+1$ and consider $\beta'=[0;(-1,-3-,3)^k,-1]$. We will show that we can write $\alpha-2$ as $\alpha-2=[0;(-1,-3,-3)^k,-1,\ldots]$. Using the recurrent formula's (9) and (10) we find for $0\leq j\leq k$,

$$\begin{array}{ll}
p_{3j-1}(\beta')=1-4j & q_{3j-1}(\beta')=2j\\
p_{3j}(\beta')=-8j & q_{3j}(\beta')=4j+1\\
p_{3j+1}(\beta')=-1-4j & q_{3j+1}(\beta')=2j+1
\end{array}$$

One can prove this by induction. Let $\beta=[0;(-1,-3,-3)^k,-1,-5,-1,-(2k+3)]$. We will show that $\beta=\alpha-2$. The numerators and denominators of the convergents of β satisfy:

$$\begin{aligned} p_{3k}(\beta) &= -8k & q_{3k}(\beta) &= 4k+1 \\ p_{3k+1} &= -4k-1 & q_{3k+1} &= 2k+1 \\ p_{3k+2}(\beta) &= -12k-5 & q_{3k+2}(\beta) &= 6k+4 \\ p_{3k+3}(\beta) &= -8k-4 & q_{3k+3}(\beta) &= 4k+3 \\ p_{3k+4}(\beta) &= -16k^2-20k-7 & q_{3k+4}(\beta) &= 8k^2+12k+5. \end{aligned}$$

This gives

$$\beta = \frac{-16k^2-20k-7}{8k^2+12k+5}$$

and by substituting $n = 2k+1$ in $b_n - 2$, as given in (38), we find

$$b_n - 2 = \frac{-16k^2-20k-7}{8k^2+12k+5}.$$

We conclude that $\beta = b_n - 2$, so $[0; (-1,-3,-3)^k, -1, -5, -1, -(2k+3)]$ is an expansion of $b_n - 2$. In order to prove this is the odd α-continued fraction expansion of $b_n - 2$ for $\alpha = b_n$ we need to show that all points in the orbit of this continued fraction are between $\alpha - 2$ and α. Since all $\varepsilon_i(\beta) = -1$ we know that any point in the orbit is smaller than 0. Now note that to find the ordering of two continued fractions $a = [0; -a_1, -a_2, \ldots]$ and $b = [0; -b_1, -b_2, \ldots]$ we look at the first digit that differs and the ordering of these digits is taken (so if $a_i > b_i$ then $a > b$). By repeatedly deleting the first digit of $[0; (-1,-3,-3)^k, -1, -5, -1, -(2k+3)]$ and observing that the remaining continued fraction is are strictly larger than $[0; (-1,-3,-3)^k, -1, -5, -1, -(2k+3)]$ we conclude that the orbit over iterations of T_α must be larger than $\alpha - 2$. Hence, $[0; (-1,-3,-3)^k, -1, -5, -1, -(2k+3)]$ is the α-continued fraction of $b_n - 2$. In the same manner one can check the other matching indices. □

Note that we found the continued fraction expansions of α and $\alpha - 2$ as well. When doing the computations for the other sequences, this is also the case. Table 1 gives all the odd α-continued fraction expansions. The values of $\varepsilon_i(x)$ is not given in the table in order to make the table more readable. We have $\varepsilon_1(\alpha) = 1$ and $\varepsilon_i(\alpha) = -1$ for every $i \geq 2$ for every α. For $\alpha - 2$, we have that $\varepsilon_i(\alpha - 2) = -1$ for every $i \in \mathbb{N}$ and every choice of α from the table.

4.1. **Algebraic relations and corollaries thereof.** We will now show that certain algebraic relations hold for the rational values of α in our sequences. To find algebraic relations we work with Möbius transformations and matrices. Let

$$A = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}$$

be a matrix with $e_i \in \mathbb{Z}$. The Möbius transformation induced by A is the map $A : \mathbb{R} \to \mathbb{R}$ given by

$$A(z) = \frac{e_1 z + e_2}{e_3 z + e_4}. \tag{39}$$

Now let $d \in 2\mathbb{N} - 1$ and $\varepsilon = \pm 1$. We define the following matrices in $\mathrm{SL}_2(\mathbb{Z})$

$$B_{\varepsilon,d} = \begin{bmatrix} 0 & \varepsilon \\ 1 & d \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

α	digits of the odd α-CF of α	digits of the odd α-CF of $\alpha-2$	
a_n	n	$(1,3,3)^{\frac{n-1}{2}},1$	odd
	$n+1,1$	$(1,3,3)^{\frac{n-2}{2}},1,3$	even
b_n	$n+2,1,5,(1,3,3)^{\frac{n-1}{2}},1$	$(1,3,3)^{\frac{n-1}{2}},1,5,1,n+2$	odd
	$n+1,(3,3,1)^{\frac{n}{2}}$	$(1,3,3)^{\frac{n}{2}},n+1$	even
c_n	$n+2,(1,3,3)^{\frac{n-1}{2}},3,1,3,3,1$	$(1,3,3)^{\frac{n-1}{2}},1,n+2,5$	odd
	$n+1,3,(1,3,3)^{\frac{n-2}{2}},3,1,3,3,1$	$(1,3,3)^{\frac{n-2}{2}},1,3,n+1,5$	even
d_n	$n+2,(1,3,3)^{\frac{n-1}{2}},1$	$(1,3,3)^{\frac{n-1}{2}},1,n+2$	odd
	$n+1,3,(1,3,3)^{\frac{n-2}{2}},1$	$(1,3,3)^{\frac{n-2}{2}},1,3,n+1$	even

TABLE 1. The digits of the odd α-continued fractions expansions of α and $\alpha-2$ for α from the sequences of (38), but without the signs.

Fix α and $x \in [\alpha-2,\alpha)$, and let

$$M_{\alpha,x,n} = B_{\varepsilon_{\alpha,1}(x),d_{\alpha,1}(x)} B_{\varepsilon_{\alpha,2}(x),d_{\alpha,2}(x)} B_{\varepsilon_{\alpha,3}(x),d_{\alpha,3}(x)} \cdots B_{\varepsilon_{\alpha,n}(x),d_{\alpha,n}(x)}.$$

Then

$$M_{\alpha,x,n} = \begin{bmatrix} p_{\alpha,n-1}(x) & p_{\alpha,n}(x) \\ q_{\alpha,n-1}(x) & q_{\alpha,n}(x) \end{bmatrix}. \tag{40}$$

Lemma 4.3. *For all α from the sequences of (38) with (N,M) minimal such that $T_\alpha^N(\alpha) = T_\alpha^M(\alpha-2) = 0$ we have*

$$M_{\alpha,\alpha,N} = R^2 M_{\alpha,\alpha-2,M} VSR^2S. \tag{41}$$

Equation (41) will turn out to be very helpful later. For Nakada's α-continued fractions, a similar equation holds for the analogous rational points in that setting. In that case, the equation is given by $M_{\alpha,\alpha,N} = RM_{\alpha,\alpha-2,M}VSRS$; see [6]. Note that $R(x) = x+1$ and that in our case the endpoints of the interval on which we defined T_α are α and $\alpha-2$ instead of α and $\alpha-1$.

Proof. Let $\alpha = \frac{p}{q}$, with $p,q \in \mathbb{N}$ relatively prime, and $\alpha \in \{a_n, b_n, c_n, d_n\}$ for some $n \in \mathbb{N}$, $n \geq 3$. Furthermore, let (N,M) be minimal such that $T_\alpha^N(\alpha) = T_\alpha^M(\alpha-2) = 0$. From Table 1, we observe that these exponents are the matching exponents. We have that $\frac{p}{q} = \frac{p_{\alpha,N}(\alpha)}{q_{\alpha,N}(\alpha)}$ and $\frac{p}{q} - 2 = \frac{p_{\alpha,M}(\alpha-2)}{q_{\alpha,M}(\alpha-2)}$. This gives

$$M_{\alpha,\alpha,N} = \begin{bmatrix} p_{\alpha,N-1}(\alpha) & p \\ q_{\alpha,N-1}(\alpha) & q \end{bmatrix}, \quad \text{and} \quad M_{\alpha,\alpha-2,M} = \begin{bmatrix} p_{\alpha,M-1}(\alpha-2) & p-2q \\ q_{\alpha,M-1}(\alpha-2) & q \end{bmatrix}.$$

To be able to verify (41) we need the second to last convergent of α and $\alpha-2$. By using the continued fractions in Table 1 and the recurrence relations (9) and (10), we can compute these second to last convergent of α and $\alpha - 2$. This results in Table 2.

α	$M_{\alpha,\alpha,N}$	$M_{\alpha,\alpha-2,M}$	
a_n	$\begin{bmatrix} 0 & 1 \\ 1 & n \end{bmatrix}$	$\begin{bmatrix} 4-4n & 1-2n \\ 2n-1 & n \end{bmatrix}$	odd
	$\begin{bmatrix} 1 & 1 \\ n+1 & n \end{bmatrix}$	$\begin{bmatrix} 3-2n & 1-2n \\ n-1 & n \end{bmatrix}$	even
b_n	$\begin{bmatrix} 4n & 2n+1 \\ 4n^2+2n+1 & 2n^2+2n+1 \end{bmatrix}$	$\begin{bmatrix} -4n & -4n^2-2n-1 \\ 2n+1 & 2n^2+2n+1 \end{bmatrix}$	odd and even
c_n	$\begin{bmatrix} 9n-2 & 5n-1 \\ 9n^2-2n+9 & n(5n-1)+5 \end{bmatrix}$	$\begin{bmatrix} -2n^2+n-2 & 7n-10n^2-11 \\ n^2+1 & n(5n-1)+5 \end{bmatrix}$	odd and even
d_n	$\begin{bmatrix} 2n-1 & n \\ 2n^2-n+2 & n^2+1 \end{bmatrix}$	$\begin{bmatrix} 1-2n & n-2n^2-2 \\ n & n^2+1 \end{bmatrix}$	odd and even

TABLE 2. $M_{\alpha,\alpha,N}$ and $M_{\alpha,\alpha-2,M}$ for the sequences from (38). These are found by calculating the convergents using Table 1.

By substituting these matrices into (41), we find that the equation indeed holds for any α from the sequences of (38). □

The following lemma is an adaptation of Corollary 3.2.5 in [17].

Lemma 4.4. *If (41) holds for $\alpha = \frac{p}{q}$, and (N, M) are the matching exponents, then matching holds in a sufficiently small neighborhood of α with matching exponents $(N+1, M+1)$. Furthermore, let x be a point in such neighborhood of α. Then*

$$M_{x,x,N+1} = R^2 M_{x,x-2,M+1} \tag{42}$$

holds for x.

Proof. Fix $\alpha \in \mathbb{Q} \cap (0,1)$ such that (41) holds for α with matching exponents (N, M), and x in a sufficiently small neighborhood of α. Using (39) and (40) we can write x as

$$x = M_{x,x,N}(T_x^N(x)) \tag{43}$$

and

$$x = R^2 M_{x,x-2,M}(T_x^M(x-2)). \tag{44}$$

Since x is in a sufficiently small neighborhood of α we have that

$$M_{x,x,N} = M_{\alpha,\alpha,N} \text{ and } M_{x,x-2,M} = M_{\alpha,\alpha-2,M}.$$

Equations (41), (43) and (44) give

$$VSR^2ST_x^N(x) = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} \left(T_x^N(x)\right) = T_x^M(x-2).$$

Thus,

$$T_x^M(x-2) = \frac{-T_x^N(x)}{2T_x^N(x)+1}, \tag{45}$$

which implies

$$T_x^N(x) = \frac{-T_x^M(x-2)}{2T_x^M(x-2)+1}.$$

Now

$$\begin{aligned} T_x^{M+1}(x-2) &= \left|\frac{2T_x^N(x)+1}{-T_x^N(x)}\right| - d_{x,M+1}(x-2), \\ T_x^{N+1}(x) &= \left|\frac{2T_x^M(x-2)+1}{-T_x^M(x-2)}\right| - d_{x,N+1}(x). \end{aligned}$$

This gives

$$T_x^{M+1}(x-2)-T_x^{N+1}(x) = \left|\frac{-1}{T_x^N(x)}-2\right| - \left|\frac{-1}{T_x^M(x-2)}-2\right| + d_{x,N+1}(x) - d_{x,M+1}(x-2).$$

Using (45) we find

$$T_x^{M+1}(x-2)-T_x^{N+1}(x) = \left|\frac{-1}{T_x^N(x)}-2\right| - \left|\frac{1}{T_x^N(x)}\right| + d_{x,N+1}(x) - d_{x,M+1}(x-2).$$

When $T_x^N(x) > 0$, we find

$$T_x^{M+1}(x-2)-T_x^{N+1}(x) = \frac{1}{T_x^N(x)} + 2 - \frac{1}{T_x^N(x)} + d_{x,N+1}(x) - d_{x,M+1}(x-2).$$

When $T_x^N(x) < 0$, since x is in a neighborhood of α, $T_x^N(x)$ is in the neighborhood of 0, and so

$$T_x^{M+1}(x-2)-T_x^{N+1}(x) = -\frac{1}{T_x^N(x)} - 2 + \frac{1}{T_x^N(x)} + d_{x,N+1}(x) - d_{x,M+1}(x-2).$$

In both cases, due to the fact that the digits are odd, we have

$$T_x^{N+1}(x) - T_x^{M+1}(x-1) = r$$

for some $r \in 2\mathbb{Z}$. Since $T_x^{N+1}(x), T_x^{M+1}(x-2) \in [x-2, x)$ we find $r \in 2\mathbb{Z} \cap (-2,2) = \{0\}$, and therefore we have $T_x^{N+1}(x) = T_x^{M+1}(x-1)$. Furthermore, since $x = M_{x,x,N+1}(T_x^{N+1}(x))$ and $x = R^2 M_{x,x-2,M+1}(T_x^{M+1}(x-2))$ we have that (42) holds. □

For matching to hold we do not need the assumption of ergodicity. However, in order to prove monotonicity of the entropy function on a matching interval this assumption is required. We also need the exponential convergence of the continued fraction algorithm for almost every x. Under the assumption of ergodicity and given that the entropy is positive for every $\alpha > 0$ we have such convergence using the Shannon-McMillan-Breiman-Chung Theorem. Positive entropy is observed by the numerics. The Shannon-McMillan-Breiman-Chung Theorem gives us that the entropy can be found by fixing a typical x of the dynamical system and looking at the exponential shrinking rate of the cylinders containing x. The size of the n^{th} order cylinder containing x is given by $\frac{\alpha}{(q_n(x)+\alpha q_{n-1}(x))(q_n(x)+(\alpha-2)q_{n-1}(x))}$ which is analogous to α-continued fractions (see [20]). Observe that the exponential shrinking rate can only be positive if q_n grows exponentially fast.

Proof of Theorem 4.1. Note that in Lemma 4.3 and 4.4, we found a matching interval for every α from the sequences of (38) for which equation (42) hold for all elements in this interval. This equation is necessary and sufficient to prove that on such interval, if $N - M < 0$, the entropy is increasing; if $N - M = 0$, the entropy is constant; and if $N-M>0$, the entropy is decreasing. The proof is the same as for the other families mentioned with the only adaptation that $\alpha-1$ is replaced by $\alpha-2$; see [17], Section 3. We give a short sketch of the proof.

We will prove that for α,α' on a matching interval with $\alpha'<\alpha$ that the following formula holds

$$\frac{h(\alpha)}{h(\alpha')}=1+(M-N)\mu_\alpha([\alpha',\alpha]) \tag{46}$$

where μ_α is the invariant measure for T_α. Since $\alpha'<\alpha$, we conclude that:

- if $N-M<0$, $\frac{h(\alpha)}{h(\alpha')}=1+(M-N)\mu_\alpha([\alpha',\alpha])<1$. That is, $h(\alpha)<h(\alpha')$ and entropy is increasing,
- if $N-M=0$, $\frac{h(\alpha)}{h(\alpha')}=1+(M-N)\mu_\alpha([\alpha',\alpha])=1$. That is, $h(\alpha)=h(\alpha')$ and entropy is constant,
- if $N-M>0$, $\frac{h(\alpha)}{h(\alpha')}=1+(M-N)\mu_\alpha([\alpha',\alpha])>1$. That is, $h(\alpha)>h(\alpha')$ and entropy is decreasing.

In order to show this, we choose a point $x\in(\alpha',\alpha)$ such that x is a typical point for T_α and $x-2$ is a typical point for $T_{\alpha'}$, that is Birkhoff's Ergodic Theorem holds for both x and $x-2$. The points x and $x-2$ match before x returns to the interval (α',α). Let n_1 be the minimal positive integer such that $T_\alpha^{n_1}(x)\in(\alpha',\alpha)$, which we call the *first return time to* (α',α). Let m_1 be the return time of $x-2$ to $(\alpha'-2,\alpha-2)$ then $n_1-m_1=N-M$ and $q_{\alpha,n_1-1}(x)=q_{\alpha',n_1-1}(x-2)$. Now, since x and $x-2$ are typical points, so are $T_\alpha^{n_1}(x)$ and $T_\alpha^{m_1}(x-2)$. Furthermore, the distance between the two points is exactly 2 and therefore we are in the initial situation again. This process repeats again so that for the kth return time of x and $x-2$ we have $n_k-m_k=(N-M)k$ and also $q_{\alpha,n_k-1}(x)=q_{\alpha',m_k-1}(x-2)$. From Birkhoff's Ergodic Theorem it follows that $\lim_{n\to\infty}\frac{k}{n_k}=\mu_\alpha([\alpha',\alpha])$ and together with the formula

$$h(\alpha)=2\lim_{n\to\infty}\frac{1}{n}\log(q_{n,\alpha}(x)) \tag{47}$$

one can derive (46). Note that heuristically, we say that when the entropy is low we used a lot of insertions in constructing the continued fraction of a typical x and when the entropy is high we used a lot of singularisations. From (46) one can see how the matching index influences the change in entropy. Now from Proposition 4.2 we find that for $\alpha=a_n$ we have $N-M<0$, for $\alpha=b_n$ and $\alpha=d_n$ we have $N-M=0$ and for $\alpha=c_n$ we have $N-M>0$. This finishes the proof. □

An interesting observation is that, in order to prove the same statement for Nakada's α-continued fractions, one can use the same sequences. Though one can check that we do not have the same intervals of monotonicity of the entropy for both families. The following theorem characterises the larger matching intervals on what we believe to be the largest entropy plateaux, $[g^2,G]$. As we later shall see, even on the interval $[g^2,G]$ we observe a rich structure of matching intervals. This phenomena is not exhibited by the other continued fraction algorithms mentioned. Furthermore, the theorem provides us that the largest entropy plateaux cannot be larger than $[g^2,G]$.

Theorem 4.5. *For* $n\in\mathbb{N}$, *the intervals* $I_n:=([0;2,\overline{1^{2n},3}],[0;\overline{2,1^{2n-2},2}])$ *are matching intervals with matching exponents* $(n+3,n+3)$.[5] *Furthermore, the left convergents* $l_{2n+1}=[0;2,1^{2n+1}]$ *of* g^2 *match with matching exponents* $(n+1,n+4)$.

[5] By $[0;\overline{2,1^0,3}]$ we mean $[0;\overline{2,3}]$.

Proof. Let $r_{2n} = [0; 2, 1^{2n}]$ be the right approximants of g^2. Then the Odd α-continued fraction for $\alpha = r_{2n}$ of r_{2n} is given by $r_{2n} = [0; 3, (-3)^n, -1]_{r_{2n}}$, and for $r_{2n}-2$, we have $r_{2n}-2 = [0; -1, (-3)^{n+1}]_{r_{2n}}$. We find that for r_n, we have matching exponents $(n+2, n+2)$. The proof of these expansions goes similar to the proof of the sequences in Proposition 4.2. In the same manner as Lemma 4.3, we can show that Equation (41) holds which gives us that there must be a matching interval containing r_{2n}. We will now determine the boundaries of this interval precisely. We want to find α close enough to r_{2n} such that we use the same branches when iterating α over T_α as iterating r_{2n} over $T_{r_{2n}}$. If we take the same prefix for α as for r_{2n}, it is easy to ensure that $\alpha = [0; 3, (-3)^n, \ldots]$. We have to ensure that the digit on place $n+2$ equals to 1. We also have to ensure that $\alpha-2 = [0; -1, (-3)^n, \ldots]$ and that the digit on place $n+2$ equals to 3, see Figure 14. When $\alpha < r_{2n}$ we find the conditions : $T_\alpha^{n+1}(\alpha) < \frac{-1}{\alpha+1}$ and $T_\alpha^{n+1}(\alpha-2) > \frac{-1}{\alpha+1}$ of which $T_\alpha^{n+1}(\alpha) < \frac{-1}{\alpha+1}$ is most constraining giving that the left endpoint of the interval is $\alpha = [0; 3, (-3)^n, \frac{-1}{\alpha+1}] = [0; 2, \overline{1^{2n}, 3}]$ and when $\alpha > r_{2n}$ we find the conditions $T_\alpha^{n+1}(\alpha) > \alpha-2$ and $T_\alpha^{n+1}(\alpha-2) < \frac{-1}{\alpha+3}$ of which $T_\alpha^{n+1}(\alpha-2) < \frac{-1}{\alpha+3}$ is most constraining giving that the right endpoint of the interval satisfies $\alpha - 2 = [0; -1, (-3)^n, \frac{-1}{\alpha+3}]$ which gives $\alpha = [0; \overline{2, 1^{2n-2}, 2}]$.

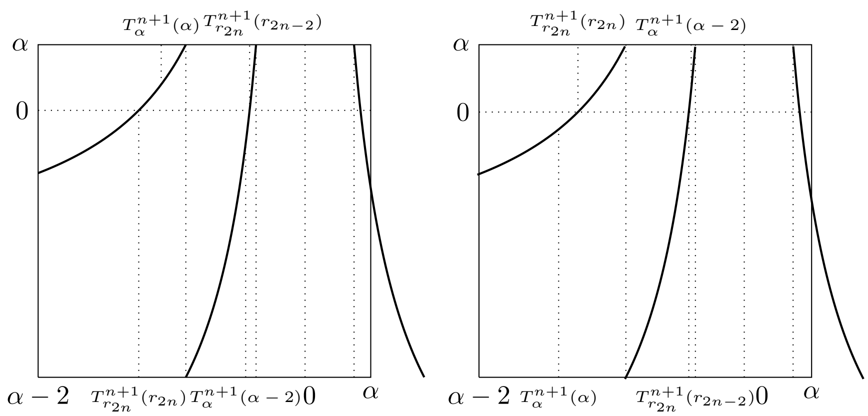

FIGURE 14. Left: $\alpha < r_{2n}$, right: $\alpha > r_{2n}$. Note that in both cases the ordering of α and r_{2n} swaps but the ordering of $\alpha-2$ and $r_{2n}-2$ remains preserved.

Now for l_{2n+1} we find $l_{2n+1} = [0; 3, (-3)^n]_{l_{2n+1}}$ and $l_{2n+1}-2 = [-1; (-3)^{n+2}, -1]_{l_{2n+1}}$, and so we have matching with exponents $(n+1, n+4)$. For l_{2n+1} we find that Equation (41) also holds. This gives us that there are intervals arbitrarily close on the left of g^2 on which the entropy is increasing. This finishes the proof. □

Note that the intervals in Theorem 4.5 do not cover $[g^2, G]$. Numerics show that the Lebesgue measure of the part that is not covered by these intervals is strictly less than 0.0246. Even though these regions are small, further inspection of these not-covered intervals reveal a very rich behavior. We investigated the largest three not-covered intervals. The behavior on all three is similar. We determined the matching exponents and which algebraic relations hold for all rational numbers with a denominator smaller than 10.000. For all of them either (41) or (42) holds and the matching index is always 0. However, if you look at the *matching exponents* you

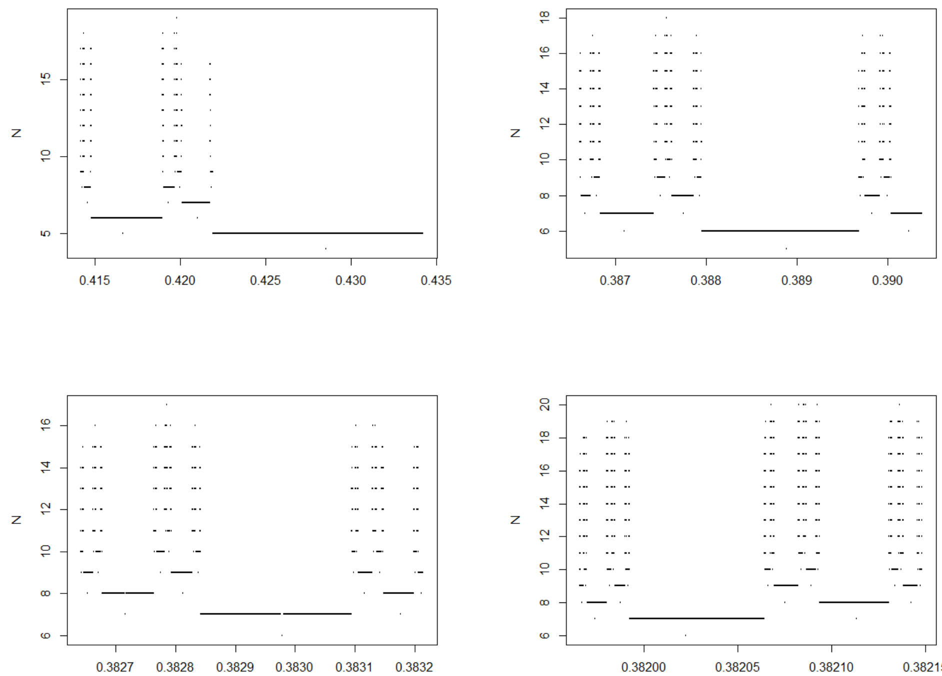

FIGURE 15. The matching exponents on the largest not-covered intervals with the matching exponents (N, N) on the y-axis. The bottom right picture is the interval from g^2 to the left end point of the interval I_3 for which we took rationals with denominator smaller than 100.000, the other pictures are for rationals with denominator smaller than 10.000.

will find that they are not on a single matching interval but rather on many; see Figure 15.

In Figure 15 we find matching exponents as high as 20. Though, with a bit more effort, one can also find rationals with higher matching exponents such as $\frac{21964041}{52071130}$ with matching exponents $(32, 32)$. If we check the algebraic relation for all rationals in $(0, g^2)$ with denominator smaller than 1.000 then we find again that either (41) or (42) holds. This, and everything else mentioned in this section strongly suggests the following conjecture to be true.

Conjecture 4.6. *Matching holds for Lebesgue almost every $\alpha \in [0, G]$. Furthermore, the largest entropy plateaux is given by $[g^2, G]$.*

Even though it would surprise us if the conjecture would be false, we were unable to prove it at this point. On the other hand, when following the same lines as in the proofs of Ito and Tanaka's α-continued fractions (or the other two families, the proofs go the same) we were unable to exclude other possible algebraic relations. Even if we can prove that only these two algebraic relations can hold we would be able to conclude that matching holds on an open dense set and it would not give us that matching holds for almost every α. In order to prove this a more precise description of the set of α for which matching does not hold might be needed. Such a description is known for the other families mentioned earlier in the paper.

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