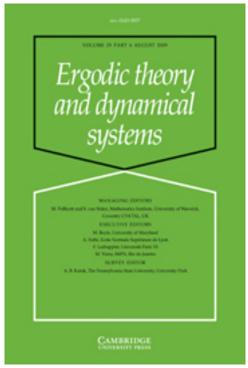

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Mixing properties of (α, β)-expansions

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Abstract. Let $0 < \alpha < 1$ and $\beta > 1$. We show that every $x \in [0, 1]$ has an expansion of the form

$$x = \sum_{n=1}^{\infty} \frac{h_n}{\beta^{\sum_{i=1}^{n} p_i} \alpha^{n - \sum_{i=1}^{n} p_i(x)}},$$

where $h_i = h_i(x) \in \{0, \alpha/\beta\}$, and $p_i = p_i(x) \in \{0, 1\}$. We study the dynamical system underlying this expansion and give the density of the invariant measure that is equivalent to the Lebesgue measure. We prove that the system is weakly Bernoulli, and we give a version of the natural extension. For special values of α, we give the relationship of this expansion with the greedy β-expansion.

1. *Introduction*

In 1957, Rényi introduced in [**R2**] a generalization of the continued fraction algorithm; the so-called f-expansions. The metrical properties of these f-expansions were investigated, and Rényi gave important results on the existence and properties of the density of the invariant measure, and conditions when the underlying system is ergodic. In the last section of [**R2**] Rényi discussed an example at length that he had introduced slightly earlier in [**R1**]. These are the β-expansions, for which the 'generating' map T_β, for $\beta > 1$, is given by

$$T_\beta(x) = \beta x (\text{mod } 1), \quad \text{for } x \in [0, 1). \tag{1}$$

Using T_β, one can show that every $x \in [0, 1)$ has a series expansion of the form

$$x = \sum_{n=1}^{\infty} \frac{a_n}{\beta^n}, \tag{2}$$

where $a_n \in \{0, 1, \ldots, \lfloor\beta\rfloor\}$ in the case $\beta \notin \mathbb{N}$, and $a_n \in \{0, 1, \ldots, \beta - 1\}$ in the case $\beta \in \mathbb{N}$.

There is a dramatic difference between the case that $\beta \in \mathbb{N}$ (in this case the T_β-invariant measure is Lebesgue measure λ on $[0, 1)$, and the digits are independent, i.e. the underlying dynamical system is Bernoulli), and the case that $\beta \notin \mathbb{N}$. In this last case, the Lebesgue measure is certainly not the T_β-invariant measure. In fact, Rényi showed that in this case the density h_β of the T_β-invariant measure is bounded by $\ell = 1 - 1/\beta$ and $h = 1/\ell$, and that the underlying system is ergodic. He was also able to find the density of the T_β-invariant measure, in the case where β is equal to the *golden mean* $G = (1/2)(\sqrt{5} + 1)$. Shortly afterwards Gel'fond (in [**Gel**]) and Parry (in [**Par**]) independently obtained an exact expression for the density h_β.

Expansions to base β have provided a wide and deep field for research, toy models, etc, exactly because there is such a difference in behavior of the map T_β when β is an integer or not. For example, in the case where $\beta \geq 2$ is an integer, only certain rationals have a finite expansion, while in the case where β is not an integer, almost every $x \in [0, 1)$ has uncountably many different series expansions of the form (2).

There are also a number of interesting variations on β-expansions. For example, in [**W1**], Wilkinson considered so-called (α, β)-expansions, for which the 'generating' map is given by $T_{\alpha,\beta}(x) = \beta x + \alpha (\text{mod } 1)$, and shows that for $\beta > 2$ the underlying dynamical systems are weakly Bernoulli. The more difficult situation $1 < \beta < 2$ was investigated in [**Pal**]; see also [**FL**]. Another interesting generalization was given recently by Góra in [**Go**].

Although there are many papers on piecewise linear maps, where the multiplication factor in each case is greater than $1 + \varepsilon$ for some $\varepsilon > 0$ (see e.g. [**W2, K, Ry**]), relatively few papers exist where the map is expanding on at least one branch and contracting on at least one other branch; see e.g. [**BF, CLdR, I**].

In this paper, we study another kind of (α, β)-expansions based on piecewise linear maps T, which are (like the Wilkinson–Palmer map $T_{\alpha,\beta}$) variations on the map T_β as defined in (1). The big difference here is that T is expanding on one branch and contracting on another branch.

1.1. *(α, β)-expansions.* Let $0 < \alpha < 1$ and $1 < \beta < 2$. Consider the transformation $T : [0, 1] \to [0, 1]$, given by

$$T(x) = \begin{cases} \beta x, & x \in [0, 1/\beta) = I_0, \\ \dfrac{\alpha}{\beta}(\beta x - 1), & x \in [1/\beta, 1] = I_1, \end{cases} \tag{3}$$

see also Figure 1.

For $x \in [0, 1]$ we set

$$p = p(x) = \begin{cases} 1, & x \in I_0 \\ 0, & x \in I_1 \end{cases} \quad \text{and} \quad h = h(x) = \begin{cases} 0, & x \in I_0, \\ \dfrac{\alpha}{\beta}, & x \in I_1. \end{cases}$$

Clearly,

$$T(x) = \beta^{p(x)} \alpha^{1-p(x)} x - h(x).$$

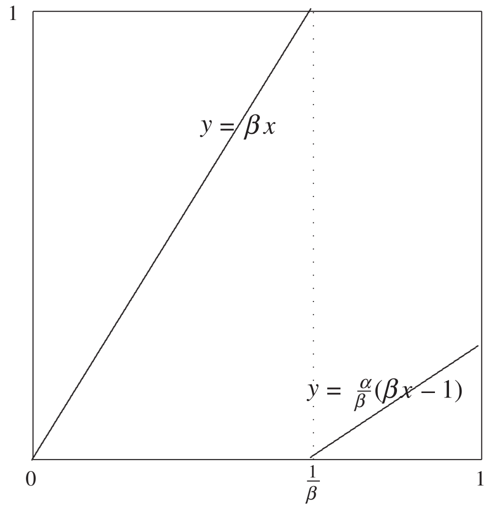

FIGURE 1. The map T.

For $n \geq 1$, define $p_n(x) = p(T^{n-1}(x))$, and $h_n(x) = h(T^{n-1}(x))$. Then, if $T^n(x) \neq 0$, we have

$$\begin{aligned}
x &= \frac{h_1(x)}{\beta^{p_1(x)}\alpha^{1-p_1(x)}} + \frac{T(x)}{\beta^{p_1(x)}\alpha^{1-p_1(x)}} \\
&= \frac{h_1(x)}{\beta^{p_1(x)}\alpha^{1-p_1(x)}} + \frac{h_2(x)}{\beta^{p_1(x)+p_2(x)}\alpha^{2-(p_1(x)+p_2(x))}} + \frac{T^2(x)}{\beta^{p_1(x)+p_2(x)}\alpha^{2-(p_1(x)+p_2(x))}} \\
&\vdots \\
&= \frac{h_1(x)}{\beta^{p_1(x)}\alpha^{1-p_1(x)}} + \cdots + \frac{h_n(x)}{\beta^{\sum_{i=1}^n p_i(x)}\alpha^{n-\sum_{i=1}^n p_i(x)}} + \frac{T^n(x)}{\beta^{\sum_{i=1}^n p_i(x)}\alpha^{n-\sum_{i=1}^n p_i(x)}}.
\end{aligned}$$

Thus, we see that if, for some m, $T^m(x) = 0$ and m is the least positive integer with this property, then x has a finite expansion of the form

$$x = \frac{h_1(x)}{\beta^{p_1(x)}\alpha^{1-p_1(x)}} + \frac{h_2(x)}{\beta^{p_1(x)+p_2(x)}\alpha^{2-(p_1(x)+p_2(x))}} + \cdots + \frac{h_m(x)}{\beta^{\sum_{i=1}^m p_i(x)}\alpha^{m-\sum_{i=1}^m p_i(x)}}.$$

Suppose now that $T^n(x) \neq 0$ for all $n \geq 1$. We claim that in this case

$$x = \sum_{n=1}^{\infty} \frac{h_n(x)}{\beta^{\sum_{i=1}^n p_i(x)}\alpha^{n-\sum_{i=1}^n p_i(x)}}. \tag{4}$$

In order to prove this claim, note that, since $T^n(x) \in [0, 1]$, it suffices to show that

$$\lim_{n\to\infty} \frac{1}{\beta^{\sum_{i=1}^n p_i(x)}\alpha^{n-\sum_{i=1}^n p_i(x)}} = 0. \tag{5}$$

For this, we show that

$$\sum_{n=1}^{\infty} \frac{1}{\beta^{\sum_{i=1}^n p_i(x)}\alpha^{n-\sum_{i=1}^n p_i(x)}} < \infty. \tag{6}$$

We have the following lemma.

LEMMA 1.1. *Let* $0 < \alpha < 1$ *and* $1 < \beta < 2$*, and let the map* T *be defined as in (3). Then we have that:*

(i) $T(I_1) \subset I_0$;

(ii) *let* $k = k(\alpha)$ *be the unique non-negative integer for which* $(1/\beta^{k+1}) < \alpha \leq (1/\beta^k)$*, then* $T^i(I_1) \subset I_0$ *for* $1 \leq i \leq k+1$.

Proof.

(i) Note that $T(I_1) = [0, (\alpha/\beta)(\beta - 1))$. Since $\alpha(\beta - 1) < 1$, then $(\alpha/\beta)(\beta - 1) < 1/\beta$. Hence, $T(I_1) \subset [0, 1/\beta) = I_0$.

(ii) Since
$$\frac{1}{\beta^{k+1}} < \alpha \leq \frac{1}{\beta^k} \quad \text{for } k = k(\alpha),$$
it follows that
$$\frac{\alpha}{\beta}(\beta - 1) \leq \frac{1}{\beta^{k+1}}(\beta - 1) < \frac{1}{\beta^{k+1}}.$$
Thus, $T(I_1) \subset [0, 1/(\beta^{k+1}))$ and, hence,
$$T^i(I_1) \subset \left[0, \frac{1}{\beta^{k+1-(i-1)}}\right) \subset I_0 \quad \text{for all } 1 \leq i \leq k+1. \qquad \square$$

Remark 1.1. Suppose that $T^m(x) \neq 0$ for all $m \geq 1$. Let $k = k(\alpha) \geq 0$ be such, that $1/\beta^{k+1} < \alpha \leq 1/\beta^k$, then from Lemma 1.1 we have
$$\#\{0 \leq i \leq n-1 \mid T^i(x) \in I_1\} \leq \frac{n}{k+2} \quad \text{for all } n \geq 2.$$

PROPOSITION 1.1. *Suppose that for all* $m \geq 1$ *we have that* $T^m(x) \neq 0$*, then*
$$\sum_{n=1}^{\infty} \frac{1}{\beta^{\sum_{i=1}^n p_i(x)} \alpha^{n - \sum_{i=1}^n p_i(x)}} < \infty.$$

Proof. Since $\beta > 1$, while $\alpha < 1$, the above sum is the largest when $n - \sum_{i=1}^n p_i(x)$ takes its largest possible value for each $n \geq 1$. Now
$$n - \sum_{i=1}^n p_i(x) = \#\{0 \leq i \leq n-1 \mid T^i(x) \in I_1\}.$$
Since $1/\beta^{k+1} < \alpha \leq \beta^k$ for a unique $k = k(\alpha) \geq 0$, again by Lemma 1.1,
$$n - \sum_{n=1}^{\infty} p_i(x) = \#\{0 \leq i \leq n-1 \mid T^i(x) \in I_1\} \leq \frac{n}{k+2}.$$
Since we now have that $\alpha\beta^{k+1} > 1$, we find that
$$\sum_{n=1}^{\infty} \frac{1}{\beta^{\sum_{i=1}^n p_i(x)} \alpha^{n - \sum_{i=1}^n p_i(x)}} \leq \sum_{n=1}^{\infty} \frac{1}{\beta^{\frac{n(k+1)}{k+2}} \alpha^{\frac{n}{k+2}}} = \sum_{n=1}^{\infty} \frac{1}{(\alpha\beta^{k+1})^{\frac{n}{k+2}}} < \infty.$$
Thus,
$$\sum_{n=1}^{\infty} \frac{1}{\beta^{\sum_{i=1}^n p_i(x)} \alpha^{n - \sum_{i=1}^n p_i(x)}} < \infty. \qquad \square$$

In this paper the metrical properties of (α, β)-expansions are investigated. In particular, we show that the underlying systems are weakly Bernoulli and we also find the entropy of this dynamical system. In §5 we consider the special case where $\alpha = 1/\beta^k$, for $k \in \mathbb{N}$. In these cases the (α, β)-expansions yield 'slow' β-expansions. In fact, we see that the series expansions yielded by these (α, β)-expansions are identical to the series expansions given by the corresponding β-expansion, but that the series expansion is yielded 'in a slow way'.

In the last section the case $\beta \geq 2$ is considered. For these values of β there are two meaningful ways to define (α, β)-expansions. In the first way one defines the map T as in (3). In this case, the proof that the expansion converges, i.e. that (4) holds in the case $\beta \geq 2$, is slightly more involved than the above proof of (4) for $1 < \beta < 2$. Another way of defining (α, β)-expansions in the case $\beta \geq 2$ is along the lines of the classical β-expansion. In both cases the underlying dynamical systems are weakly Bernoulli. Since the proofs of these results are similar to the case that $1 < \beta < 2$, only outlines of these proofs are given.

2. *Digits and fundamental intervals*

2.1. *Digits.* We have seen in (4) that every $x \in [0, 1]$ can be written as†

$$x = \sum_{n=1}^{\infty} \frac{h_n}{\beta^{\sum_{i=1}^n p_i} \alpha^{n - \sum_{i=1}^n p_i}}$$

where this sum is finite if $T^m(x) = 0$ for some $m \geq 1$.

Note that the h_i and p_i are determined once we know in which element of the partition $\{I_0, I_1\}$ the point $T^{i-1}(x)$ lies. Define for $x \in [0, 1]$ the sequence of digits $a_n = a_n(x)$, $n \geq 1$, by

$$a_n = k \quad \text{if and only if } T^{n-1}(x) \in I_k, \text{ where } k \in \{0, 1\}. \tag{7}$$

We call the sequence $(a_n)_{n\geq 1}$ the (α, β)-digits of x. Note that the sequence $(a_n)_{n\geq 1}$ completely determines the expression (4) and vice versa. So we identify x with its sequence of (α, β)-digits,

$$x = \sum_{n=1}^{\infty} \frac{h_n}{\beta^{\sum_{i=1}^n p_i} \alpha^{n - \sum_{i=1}^n p_i}} =: [a_1, a_2, \ldots].$$

In fact, since for $n \geq 1$, $a_n = 1 - p_n$, and by the definition of h_n we have that

$$x = \sum_{n=1}^{\infty} \frac{a_n}{\beta^{n+1-\sum_{i=1}^n a_i} \alpha^{-1+\sum_{i=1}^n a_i}}.$$

2.2. *Fundamental intervals.* We define fundamental intervals (of rank n) in the usual way: the intervals of rank one are $\Delta(i) = \{x \mid a_1(x) = i\} = I_i$, for $i \in \{0, 1\}$, and the intervals of rank n, for $n \geq 2$ are

$$\begin{aligned}\Delta(i_1, \ldots, i_n) &= \Delta(i_1) \cap T^{-1}\Delta(i_2) \cap \cdots \cap T^{-(n-1)}\Delta(i_n) \\ &= \{x \mid a_1(x) = i_1, \ldots, a_n(x) = i_n\} \\ &= \left\{x \mid x = \frac{h_1}{\beta^{p_1}\alpha^{1-p_1}} + \cdots + \frac{h_n}{\beta^{\sum_{i=1}^n p_i}\alpha^{n-\sum_{i=1}^n p_i}} + \frac{T^n(x)}{\beta^{\sum_{i=1}^n p_i}\alpha^{n-\sum_{i=1}^n p_i}}\right\},\end{aligned}$$

† We drop the argument whenever possible.

where

$$h_j = \begin{cases} 0, & i_j = 0 \\ \dfrac{\alpha}{\beta}, & i_j = 1 \end{cases} \quad \text{and} \quad p_j = \begin{cases} 1, & i_j = 0, \\ 0, & i_j = 1. \end{cases}$$

On $\Delta(i_1, \ldots, i_n)$, the map T^n is linear with slope $\beta^{\sum_{i=1}^n p_i} \alpha^{n-\sum_{i=1}^n p_i}$.

A fundamental interval $\Delta(i_1, \ldots, i_n)$ is *full* if $\lambda(T^n(\Delta(i_1, \ldots, i_n))) = 1$. Here λ denotes Lebesgue measure on [0, 1]. From the above we see that if $\Delta(i_1, \ldots, i_n)$ is full, it is equal to the interval

$$\left[\sum_{m=1}^{n} \frac{h_m}{\beta^{\sum_{i=1}^m p_i} \alpha^{m-\sum_{i=1}^m p_i}}, \sum_{m=1}^{n} \frac{h_m}{\beta^{\sum_{i=1}^m p_i} \alpha^{m-\sum_{i=1}^m p_i}} + \frac{1}{\beta^{\sum_{i=1}^n p_i} \alpha^{n-\sum_{i=1}^n p_i}}\right),$$

and

$$\lambda(\Delta(i_1, \ldots, i_n)) = \frac{1}{\beta^{\sum_{i=1}^n p_i} \alpha^{n-\sum_{i=1}^n p_i}}. \tag{8}$$

In §2.3 we show that the full intervals generate the Borel σ-algebra.

We now consider non-full fundamental intervals that are not subsets of full intervals of lower rank. Let B_n be the collection of non-full intervals of rank n that are not subsets of full intervals of lower rank.

Note that $\Delta(1)$ is the only member of B_1 and B_2, since $\Delta(1) = \Delta(10)$. Suppose that $\Delta(i_1, \ldots, i_n)$ is an element of B_n, then $\Delta(i_1, \ldots, i_j) \in B_j$ for $1 \leq j \leq n-1$. We claim that $\Delta(i_1, \ldots, i_n)$ contains exactly one element of B_{n+1}. There are two cases:

- if $T^n(\Delta(i_1, \ldots, i_n)) \cap \Delta(1) = \emptyset$, then $\Delta(i_1, \ldots, i_n, 0) = \Delta(i_1, \ldots, i_n)$, and $\Delta(i_1, \ldots, i_n, 0)$ is the only member of B_{n+1} contained in $\Delta(i_1, \ldots, i_n)$;
- if $T^n(\Delta(i_1, \ldots, i_n)) \cap \Delta(1) \neq \emptyset$, then $\Delta(i_1, \ldots, i_n, 0)$ is full, $\Delta(i_1, \ldots, i_n, 1)$ is non-full and therefore in B_{n+1}; furthermore, $\lambda(\Delta(i_1, \ldots, i_n, 1)) < (1/\beta)$ $\lambda(\Delta(i_1, \ldots, i_n))$.

Since $|B_1| = |B_2| = 1$, it thus follows by induction from the above that $|B_n| = 1$ for all n.

Let $B_n = \{\Delta(i_1, \ldots, i_n)\}$, then it follows from the above that

$$\lambda(\Delta(i_1, \ldots, i_n)) = \lambda(\Delta(i_1, \ldots, i_{n-1})) \quad \text{if } T^{n-1}(\Delta(i_1, \ldots, i_{n-1})) \cap \Delta(1) = \emptyset,$$

and

$$\lambda(\Delta(i_1, \ldots, i_n)) < \frac{1}{\beta}\lambda(\Delta(i_1, \ldots, i_{n-1})) \quad \text{if } T^{n-1}(\Delta(i_1, \ldots, i_{n-1})) \cap \Delta(1) \neq \emptyset.$$

By induction, this implies that

$$\lambda(\Delta(i_1, \ldots, i_n)) < \frac{1}{\beta^{n-\sum_{i=1}^n p_i}},$$

where

$$n - \sum_{i=1}^{n} p_i = \#\{0 \leq j \leq n-1 \mid T^j(x) \in \Delta(1)\}$$

for any $x \in \Delta(i_1, \ldots, i_n)$. Note that, since T is expanding on $\Delta(0)$, we have that

$$\lim_{n\to\infty}\left(n - \sum_{i=1}^{n} p_i\right) = \infty.$$

2.3. *Full intervals generate the Borel* σ*-algebra.* We now show that full intervals generate the Borel σ-algebra on [0, 1]. We first introduce some notation. Let F_n be the collection of all full intervals of rank n, and let D_n be the collection of full intervals of rank n that are not subsets of full intervals of lower rank, i.e.

$$D_n = \{\Delta(i_1, \ldots, i_n) \in F_n \mid \Delta(i_1, \ldots, i_j) \notin F_j \text{ for any } 1 \leq j \leq n-1\}.$$

We have the following lemma.

LEMMA 2.1. *The union of all full intervals that are not subsets of full intervals of lower rank has full Lebesgue measure, i.e.*

$$\lambda\bigg(\bigcup_{n=1}^{\infty}\bigcup_{D_n} \Delta(i_1, \ldots, i_n)\bigg) = 1.$$

Proof. For any $N \geq 1$,

$$\begin{aligned}\lambda\bigg([0,1) \setminus \bigcup_{n=1}^{N}\bigcup_{D_n} \Delta(j_1, \ldots, j_n)\bigg) &= \lambda\bigg(\bigcup_{B_n} \Delta(i_1, \ldots, i_n)\bigg) = \lambda(\Delta(i_1, \ldots, i_n)) \\ &< \frac{1}{\beta^{N-\sum_{i=1}^{N} p_i}},\end{aligned}$$

where $\Delta(i_1, \ldots, i_n)$ is the unique element of B_n. Taking the limit as N tends to infinity, we obtain

$$\lambda\bigg([0,1) \setminus \bigcup_{n=1}^{\infty}\bigcup_{D_n} \Delta(j_1, \ldots, j_n)\bigg) = 0. \qquad \square$$

Remark 2.1. Lemma 2.1 implies that

$$\lambda\bigg(\bigcup_{n=1}^{\infty}\bigcup_{F_n} \Delta(i_1, \ldots, i_n)\bigg) = 1.$$

So applying a similar procedure to any interval, we find that any interval can be covered by a countable disjoint union of full intervals.

LEMMA 2.2. *Let* $\Delta(i_1, \ldots, i_n)$ *be the unique element of* B_n*, then*

$$T^n(\Delta(i_1, \ldots, i_n)) = [0, T^n(1)) \quad \textit{for } n \geq 1.$$

Proof. The proof proceeds by induction. First note that $B_1 = \{\Delta(1)\}$, and that $T\Delta(1) = [0, T(1))$. Furthermore, $B_2 = \{\Delta(1) = \Delta(10)\}$, so $T^2\Delta(10) = [0, T^2(1))$.

Suppose the statement holds for index n. Let $\Delta(i_1, \ldots, i_n)$ be the unique element of B_n, then by assumption $T^n\Delta(i_1, \ldots, i_n) = [0, T^n(1))$ We have the following two cases.

- If $T^n(1) \in \Delta(0)$, then $B_{n+1} = \{\Delta(i_1, \ldots, i_n, 0)\}$, and

$$T^{n+1}\Delta(i_1, \ldots, i_n, 0) = T^{n+1}\Delta(i_1, \ldots, i_n) = [0, T^{n+1}(1)).$$

- If $T^n(1) \in \Delta(1)$, then $B_{n+1} = \{\Delta(i_1, \ldots, i_n, 1)\}$ and

$$T^n\Delta(i_1, \ldots, i_n, 1) = [1/\beta, T^n(1)), \quad \text{so } T^{n+1}\Delta(i_1, \ldots, i_n, 1) = [0, T^{n+1}(1)).$$

$\square$

3. *Natural extension of T*

In recent years, the use of natural extensions has contributed greatly to the development of the theory of many number theoretical maps and algorithms; see e.g. [**BJW**], where the natural extension of the Gauss map was crucial in proving the so-called Doeblin–Lenstra conjecture, or [**DKS**], where the natural extension yields in a simple and elegant way the underlying invariant measure, which is the Parry measure. See also [**DK**] or [**IK**] for a more detailed discussion of these and other related results.

3.1. *Construction of the natural extension.* In this section we derive (a version of) the natural extension of the dynamical system underlying (α, β)-expansions. Throughout this section, for $n \geq 1$, let $p_n = p_n(1)$, $h_n = h_n(1)$ and

$$\frac{P_n}{Q_n} = \frac{h_1}{\beta^{p_1}\alpha^{1-p_1}} + \cdots + \frac{h_n}{\beta^{\sum_{i=1}^n p_i}\alpha^{n-\sum_{i=1}^n p_i}}.$$

Set $R_0 = [0, 1) \times [0, 1)$, and for $n \geq 1$, set

$$R_n = [0, T^n(1)) \times \left[0, \frac{1}{\beta^{\sum_{i=1}^n p_i}\alpha^{n-\sum_{i=1}^n p_i}}\right).$$

Furthermore, let

$$Z = \bigcup_{n=0}^{\infty} R_n \times \{n\},$$

and let $\bar{\mathcal{B}} = \bigsqcup_n \mathcal{B}_n$ be the disjoint union of the Borel σ-algebras $\mathcal{B}_n$ on $R_n \times \{n\}$.

Denoting by $\tilde{\lambda}$ the two-dimensional Lebesgue measure, we have by Proposition 1.1,

$$\sum_{n=0}^{\infty} \tilde{\lambda}(R_n) \leq 1 + \sum_{n=1}^{\infty} \frac{1}{\beta^{\sum_{i=1}^n p_i}\alpha^{n-\sum_{i=1}^n p_i}} < \infty,$$

so the Lebesgue measure $\tilde{\lambda}(Z)$ of Z is finite. Let $\bar{\lambda}$ be the normalized Lebesgue measure on Z. Now define $\mathcal{T}$ on Z as follows:

- if $(x, y, 0) \in R_0 \times \{0\}$, then

$$\mathcal{T}(x, y, 0) = \begin{cases} (T(x), y/\beta, 0), & x \in \Delta(0), \\ (T(x), y/\alpha, 1), & x \in \Delta(1); \end{cases}$$

- if $(x, y, n) \in R_n \times \{n\}$, $n \geq 1$, and $T^n(1) \in \Delta(0)$,

$$\mathcal{T}(x, y, n) = (T(x), y/\beta, n+1);$$

- if $(x, y, n) \in R_n \times \{n\}$, $n \geq 1$, and $T^n(1) \in \Delta(1)$, then for $x \in \Delta(0)$ one has,

$$\mathcal{T}(x, y, n) = \left(T(x), \frac{P_n}{Q_n} + \frac{y}{\beta}, 0\right),$$

and for $x \in \Delta(1)$ one has,

$$\mathcal{T}(x, y, n) = (T(x), y/\alpha, n+1).$$

One can also describe $\mathcal{T}$ on $R_n \times \{n\}$, $n \geq 1$, using the (α, β)-digits of 1 in the following way. Let $(d_n)_{n\geq 1}$ be the (α, β)-digits of 1 and let $(x, y, n) \in R_n \times \{n\}$. Suppose that $(a_n)_{n\geq 1}$ is the sequence of (α, β)-digits of x. Then

$$\mathcal{T}(x, y, n) = \begin{cases} \left(T(x), \dfrac{P_n}{Q_n} + \dfrac{y}{\beta}, 0\right), & a_1 < d_{n+1}, \\ \left(T(x), \dfrac{y}{\beta^{p_{n+1}}\alpha^{1-p_{n+1}}}, n+1\right), & a_1 = d_{n+1}, \end{cases}$$

From the above definition of $\mathcal{T}$ one has that

$$\mathcal{T}(R_0 \times \{0\}) = ([0, 1) \times [0, 1/\beta) \times \{0\}) \cup (R_1 \times \{1\}),$$

and, for ≥ 1,

$$\mathcal{T}(R_n \times \{n\}) = \begin{cases} R_{n+1} \times \{n+1\}, & d_{n+1} = 0, \\ \left([0, 1) \times \left[\dfrac{P_n}{Q_n}, \dfrac{P_{n+1}}{Q_{n+1}}\right) \times \{0\}\right) \cup (R_{n+1} \times \{n+1\}), & d_{n+1} = 1. \end{cases}$$

Since $\lim_{n\to\infty}(P_n/Q_n) = 1$, we see that $\mathcal{T}$ is surjective. It is easily seen that $\mathcal{T}$ is injective, measurable and Lebesgue measure preserving.

Let $\pi : Z \to [0, 1]$ be the projection on the first coordinate, and let $\mathcal{B}$ be the Borel σ-algebra on $[0, 1]$. We want to show that

$$\bar{\mathcal{B}} = \coprod_{i=0}^{\infty} \mathcal{B}_n = \coprod_{i=0}^{\infty} \bigvee_{n=0}^{\infty} \mathcal{T}^n \pi^{-1} \mathcal{B} \times \{i\}.$$

Note that $\mathcal{B}_0$ is generated by sets of the form

$$\Delta(a_1, \ldots, a_n) \times \Delta(b_1, \ldots, b_m) \times \{0\},$$

where $\Delta(a_1, \ldots, a_n)$ and $\Delta(b_1, \ldots, b_m)$ are full intervals in $[0, 1]$. We now specify a particular generator of $\mathcal{B}_n$, $n \geq 1$. For each $n \geq 1$, the map

$$\psi_n : [0, 1) \to \left[0, \frac{1}{\beta^{\sum_{i=1}^n p_i}\alpha^{n-\sum_{i=1}^n p_i}}\right),$$

given by

$$\psi_n(x) = \frac{x}{\beta^{\sum_{=1}^n p_i}\alpha^{n-\sum_{i=1}^n p_i}},$$

is a continuous isomorphism. Hence, sets of the form

$$\{\psi_n(\Delta(b_1, \ldots, b_m)) \mid \Delta(b_1, \ldots, b_m) \text{ is full}\}$$

generate the σ-algebra on

$$\left[0, \frac{1}{\beta^{\sum_{i=1}^n p_i}\alpha^{n-\sum_{i=1}^n p_i}}\right).$$

Now the Borel σ-algebra on $[0, T^n 1)$ is generated by sets of the form

$$\Delta^{(n)}(a_1, \ldots, a_k) = \Delta(a_1, \ldots, a_k) \cap [0, T^n(1)),$$

where $\Delta^{(0)}(a_1, \ldots, a_k) = \Delta(a_1, \ldots, a_k)$ is a full fundamental interval in $[0, 1)$. Thus, $\mathcal{B}_n$ is generated by sets of the form

$$\Delta^{(n)}(a_1, \ldots, a_k) \times \psi_n(\Delta(b_1, \ldots, b_m)) \times \{n\},$$

where $\Delta(a_1, \ldots, a_k)$ and $\Delta(b_1, \ldots, b_m)$ are full intervals.

Since

$$\Delta^{(n)}(a_1, \ldots, a_k) \times \psi_n(\Delta(b_1, \ldots, b_m)) \times \{n\}$$

is equal to

$$\mathcal{T}^n \Delta(d_1, \ldots, d_n, a_1, \ldots, a_k) \times \Delta(b_1, \ldots, b_m) \times \{0\},$$

we only need to show that

$$\Delta(a_1, \ldots, a_k) \times \Delta(b_1, \ldots, b_m) \times \{0\} \in \mathcal{T}^m \pi^{-1}\mathcal{B} \times \{0\}.$$

To this end, divide $b_1 \cdots b_m$ into (full) subblocks $C_1 \cdots C_\ell$ as follows. Let

$$r_1 = \inf\{j \geq 1 \mid T^j \Delta(b_1, \ldots, b_j) = [0, 1)\},$$

and set $C_1 = b_1 \cdots b_{r_1}$. Next consider $b_{r_1+1} \cdots b_m$; set

$$r_2 = \inf\{j \geq 1 \mid T^j \Delta(b_{r_1+1}, \ldots, b_{r_1+j}) = [0, 1)\},$$

and $C_2 = b_{r_1+1} \cdots b_{r_1+r_2}$. Continuing in this way, we obtain $r_1 < r_2 < \cdots < r_\ell$, such that

$$C_j = b_{r_1+\cdots+r_{j-1}+1} \cdots b_{r_1+\cdots+r_j}, \quad 1 \leq j \leq \ell,$$

$T^{r_j}\Delta(C_j) = [0, 1)$ and $\Delta(b_1, \ldots, b_m) = \Delta(C_1, \ldots, C_\ell)$.

If we consider

$$\Delta = \Delta(b_1, \ldots, b_m) \times [0, 1) \times \{0\},$$

then r_1 is the first return time of elements of Δ to $R_0 \times \{0\} = [0, 1) \times [0, 1) \times \{0\}$. So, for any $x \in \Delta(b_1, \ldots, b_m)$ and any $y \in [0, 1)$,

$$\begin{aligned} r_1(x, y, 0) = r_1 &= \inf\{j \geq 1 \mid T^j \Delta(b_1, \ldots, b_m) = [0, 1)\} \\ &= \inf\{j \geq 1 \mid \mathcal{T}^j(x, y, 0) \in R_0 \times \{0\}\}. \end{aligned}$$

From the definition of $\mathcal{T}$, we see that $b_j = d_j$ for $1 \leq j \leq r_1 - 1$ and $b_{r_1} = 0$, while $d_{r_1} = 1$. Note that

$$\mathcal{T}^{r_1}\Delta(C_1, a_1, \ldots, a_n) \times [0, 1) \times \{0\} = \Delta(a_1, \ldots, a_n) \times \Delta(C_1) \times \{0\}$$

where $C_1 = b_1 \cdots b_{r_1} = d_1 \ldots d_{r_1-1}0$, and

$$\Delta(C_1) = \left[\frac{P_{r_1-1}}{Q_{r_1-1}}, \frac{P_{r_1}}{Q_{r_1}}\right).$$

Likewise, one can define r_j as the jth return time of elements of

$$\Delta(b_1, \ldots, b_m) \times [0, 1) \times \{0\} \text{ to } R_0 \times \{0\}.$$

Then, we have for any $1 \leq j \leq \ell$,

$$b_{r_1+\cdots+r_{j-1}+1} = d_1, \ldots, b_{r_1+\cdots+r_j-1} = d_{r_j-1}$$

and $b_{r_1+\cdots+r_j} = 0$, while $d_{r_j} = 1$. Moreover,

$$\mathcal{T}^{r_1+\cdots+r_j}\Delta(C_j, \ldots, C_1, a_1, \ldots, a_n) \times [0, 1) \times \{0\}$$
$$= \Delta(a_1, \ldots, a_n) \times \Delta(C_1, \ldots, C_j) \times \{0\},$$

where $C_j = d_1 d_2 \ldots d_{r_j-1}0$, and

$$\Delta(C_1 \ldots C_j) = \left[\frac{P_{r_1+\cdots+r_j-1}}{Q_{r_1+\cdots+r_j-1}}, \frac{P_{r_1+\cdots+r_j}}{Q_{r_1+\cdots+r_j}}\right).$$

Consider

$$\tilde{\Delta} = \Delta(C_\ell, C_{\ell-1}, \ldots, C_1, a_1, \ldots, a_n) \times [0, 1) \times \{0\}.$$

Note that $\Delta(C_\ell, C_{\ell-1}, \ldots, C_1)$ and $\Delta(C_\ell, C_{\ell-1}, \ldots, C_1, a_1, \ldots, a_n)$ are both full. Then

$$\mathcal{T}^m\tilde{\Delta} = \Delta(a_1, \ldots, a_n) \times \Delta(C_1, \ldots, C_\ell) \times \{0\}$$
$$= \Delta(a_1, \ldots, a_n) \times \Delta(b_1, \ldots, b_m) \times \{0\}.$$

Thus,

$$\Delta(a_1, \ldots, a_n) \times \Delta(b_1, \ldots, b_m) \times \{0\} \in \mathcal{T}^m\pi^{-1}\mathcal{B} \times \{0\}.$$

This proves that

$$\bar{\mathcal{B}} = \bigsqcup_{i=0}^{\infty}\bigvee_{n=0}^{\infty}\mathcal{T}^n\pi^{-1}\mathcal{B} \times \{i\}.$$

Define a measure μ on $[0, 1]$ by $\mu(A) = \bar{\lambda}(\pi^{-1}(A))$. Since $\mathcal{T} \circ \pi = \pi \circ T$, we see that μ is T-invariant. Furthermore, μ is equivalent to Lebesgue measure on $[0, 1]$ with density

$$h_{\alpha,\beta}(x) = C_{\alpha,\beta}\left[\mathbf{1}_{[0,1]}(x) + \sum_{n=1}^{\infty}\frac{\mathbf{1}_{[0,T^n(1))}(x)}{\beta^{\sum_{i=1}^n p_i}\alpha^{n-\sum_{i=1}^n p_i}}\right], \tag{9}$$

where

$$C_{\alpha,\beta} = \left(1 + \sum_{n=1}^{\infty}\frac{T^n 1}{\beta^{\sum_{i=1}^n p_i}\alpha^{n-\sum_{i=1}^n p_i}}\right)^{-1}$$

is a normalizing constant. We have the following theorem.

THEOREM 3.1. *The system* $(Z, \bar{\mathcal{B}}, \bar{\lambda}, \mathcal{T})$ *is a version of the natural extension of* $([0, 1], \mathcal{B}, \mu, T)$.

3.2. *Entropy.* Although the entropy of T can be calculated from general theory, we derive the entropy of T 'by hand', using the Shannon–McMillan–Breiman theorem. We first show that T is ergodic with respect to the T-invariant measure μ as given in the previous section. The proof of ergodicity is based on a classical lemma, known as Knopp's lemma; see [**DK**].

LEMMA 3.1. (Knopp's lemma) *If B is a Lebesgue set and $\mathcal{C}$ is a class of subintervals of* $[0, 1)$*, satisfying:*

(a) *every open subinterval of* $[0, 1)$ *is at most a countable union of disjoint elements from* $\mathcal{C}$;

(b) *for all* $A \in \mathcal{C}$, $\lambda(A \cap B) \geq \gamma\lambda(A)$, *where* $\gamma > 0$ *is independent of* A;

then $\lambda(B) = 1$.

THEOREM 3.2. *The system* $([0, 1], \mathcal{B}, \mu, T)$ *is ergodic.*

Proof. Let $B \in \mathcal{B}$ be such that $T^{-1}B = B$ and $\mu(B) > 0$. We need to show that $\mu(B) = 1$. Since μ is equivalent to Lebesgue measure λ on $[0, 1]$, it is enough to show that $\lambda(B) = 1$. Let $\mathcal{C}$ be the collection of all full fundamental intervals. By Remark 2.1, $\mathcal{C}$ satisfies hypothesis (a) of Knopp's lemma. Now let $A = \Delta(i_1, \ldots, i_n)$ be a full interval. From (8), we have

$$\lambda(\Delta(i_1, \ldots, i_n)) = \frac{1}{\beta^{\sum_{i=1}^n p_i}\alpha^{n-\sum_{i=1}^n p_i}}.$$

Furthermore, T^n on A is linear with slope $\beta^{\sum_{i=1}^n p_i}\alpha^{n-\sum_{i=1}^n p_i}$. Thus,

$$\lambda(A \cap B) = \lambda(A \cap T^{-n}B) = \lambda(A)\lambda(B).$$

Therefore, hypothesis (b) of Knopp's lemma is satisfied with $\gamma = \lambda(B) > 0$. Hence, $\lambda(B) = 1$ and T is ergodic. □

THEOREM 3.3. *The entropy of* T *is given by*

$$h_\mu(T) = \mu(\Delta(0)) \log\beta + \mu(\Delta(1)) \log\alpha.$$

Proof. Since the partition $\mathcal{P} = \{\Delta(0), \Delta(1)\}$ generates the σ-algebra, i.e. $\bigvee_{i=0}^{\infty} T^{-i}\mathcal{P} = \mathcal{B}$, then by the Shannon–McMillan–Breiman theorem

$$h_\mu(T) = -\lim_{n\to\infty} \frac{\log\mu(\Delta(i_1, \ldots, i_n)(x))}{n},$$

where $\Delta(i_1, \ldots, i_n)(x)$ denotes the element of $\bigvee_{i=0}^{n-1} T^{-i}\mathcal{P}$ containing x. Let

$$D_{\alpha,\beta} = 1 + \sum_{n=1}^{\infty} \frac{1}{\beta^{\sum_{i=1}^n p_i}\alpha^{n-\sum_{i=1}^n p_i}},$$

then from (9), we have that

$$C_{\alpha,\beta}\lambda(A) \leq \mu(A) \leq C_{\alpha,\beta}D_{\alpha,\beta}\lambda(A). \tag{10}$$

Hence,

$$h_\mu(T) = -\lim_{n\to\infty} \frac{\log\lambda(\Delta(i_1, \ldots, i_n)(x))}{n}.$$

Let $m_1 < m_2 < \cdots$ be such that $\Delta(i_1, \ldots, i_{m_n})(x)$ is full, then

$$\begin{aligned} h_\mu(T) &= -\lim_{n\to\infty} \frac{\log\lambda(\Delta(i_1, \ldots, i_{m_n})(x))}{m_n} \\ &= \lim_{n\to\infty} \log\beta \frac{1}{m_n}\sum_{i=1}^{m_n} p_i(x) + \lim_{n\to\infty} \log\alpha\left(1 - \frac{1}{m_n}\sum_{i=1}^{m_n} p_i(x)\right) \\ &= \mu(\Delta(0)) \log\beta + \mu(\Delta(1)) \log\alpha, \end{aligned}$$

in the last equation, we used the fact that

$$\lim_{n\to\infty}(1/m_n)\sum_{i=1}^{m_n} p_i(x) = \mu(\Delta(0)),$$

μ-almost everywhere. □

4. *Weakly Bernoulli*

We first show that the transformation T is exact. Since full intervals generate the Borel σ-algebra on [0, 1], by a result of Rohlin [**Roh**] it is enough to show that there exists a universal constant $\gamma > 0$ such that for any full interval $\Delta(i_1, \ldots, i_n)$ and any measurable subset A of $\Delta(i_1, \ldots, i_n)$ one has

$$\mu(T^n A) \le \gamma \frac{\mu(A)}{\mu(\Delta(i_1, \ldots, i_n))}.$$

To this end, let $\Delta(i_1, \ldots, i_n)$ be a full interval of order n and A a measurable subset. On $\Delta(i_1, \ldots, i_n)$ the map T^n is linear with slope $\beta^{\sum_{i=1}^n p_i}\alpha^{n-\sum_{i=1}^n p_i}$, where

$$p_j = \begin{cases} 1, & i_j = 0,\\ 0, & i_j = 1.\end{cases}$$

Then,

$$\lambda(T^n A) = \beta^{\sum_{i=1}^n p_i}\alpha^{n-\sum_{i=1}^n p_i}\lambda(A) = \frac{\lambda(A)}{\lambda(\Delta(i_1, \ldots, i_n))}.$$

By (10), we have

$$\begin{aligned}\mu(T^n A) &\le C_{\alpha,\beta} D_{\alpha,\beta}\, \lambda(T^n A) = C_{\alpha,\beta} D_{\alpha,\beta} \frac{\lambda(A)}{\lambda(\Delta(i_1, \ldots, i_n))}\\ &\le C_{\alpha,\beta} D^2_{\alpha,\beta} \frac{\mu(A)}{\mu(\Delta(i_1, \ldots, i_n))}.\end{aligned}$$

Setting $\gamma = C_{\alpha,\beta} D^2_{\alpha,\beta}$, then $\gamma > 0$ and

$$\mu(T^n A) \le \gamma \frac{\mu(A)}{\mu(\Delta(i_1, \ldots, i_n))}.$$

Thus, T is exact, hence mixing of all orders, see [**Roh**], and by results of Islam [**I**], T is weakly Bernoulli. In fact, we show that the natural extension $\mathcal{T}$ contains an induced system which is Bernoulli. This allows us to use a theorem of Saleski [**S**] to give another proof that T is weakly Bernoulli.

Throughout the rest of this section, we use the same notation as in §2, that is, for $n \ge 1$, $p_n = p_n(1)$, $h_n = h_n(1)$ and

$$\frac{P_n}{Q_n} = \frac{h_1}{\beta^{p_1}\alpha^{1-p_1}} + \cdots + \frac{h_n}{\beta^{\sum_{i=1}^n p_i}\alpha^{n-\sum_{i=1}^n p_i}}.$$

Let $\mathcal{W}$ be the induced transformation of $\mathcal{T}$ on the set $R_0 \times \{0\}$, then $\mathcal{W}(x, y, 0) = \mathcal{T}^{n(x,y,0)}$, where

$$n(x, y, 0) = \inf\{j \ge 1 \mid \mathcal{T}^j(x, y, 0) \in R_0 \times \{0\}\}.$$

For $k \geq 1$, set $R_0^k = \{(x, y, 0) \in R_0 \times \{0\} \mid n(x, y, 0) = k\}$. If $(x, y, 0) \in R_0^k$, then $\mathcal{T}^j(x, y, 0) \in R_j \times \{j\}$ for $1 \leq j \leq k-1$, while $\mathcal{T}^k(x, y, 0) \in R_0 \times \{0\}$. From the definition of $\mathcal{T}$, one sees that for $k \geq 1$,

$$R_0^k \times \{0\} = \left[\frac{P_{k-1}}{Q_{k-1}}, \frac{P_k}{Q_k}\right) \times [0, 1) \times \{0\},$$

where P_k/Q_k as given in §3, and $P_0/Q_0 = 0$. Note that

$$\left[\frac{P_{k-1}}{Q_{k-1}}, \frac{P_k}{Q_k}\right) \neq \emptyset$$

(and, hence, $R_0^k \times \{0\} \neq \emptyset$) if and only if $T^{k-1}1 \in \Delta(1)$. Furthermore,

$$\mathcal{W}(x, y, 0) = \begin{cases} \left(T(x), \dfrac{y}{\beta}, 0\right), & (x, y, 0) \in R_0^1 \times \{0\} \\ \left(T^k(x), \dfrac{P_{k-1}}{Q_{k-1}} + \dfrac{y}{\beta^{1+\sum_{i=1}^{k-1} p_i}\alpha^{(k-1)-\sum_{i=1}^{k-1} p_i}}, 0\right) & (x, y, 0) \in R_0^k \times \{0\}, \\ & \text{and } k \geq 2. \end{cases}$$

On the interval

$$\left[\frac{P_{k-1}}{Q_{k-1}}, \frac{P_k}{Q_k}\right),$$

the map T^k is linear with

$$T^k\left(\left[\frac{P_{k-1}}{Q_{k-1}}, \frac{P_k}{Q_k}\right)\right) = [0, 1),$$

and

$$T^k(x) = \beta^{1+\sum_{i=1}^{k-1} p_i}\alpha^{(k-1)-\sum_{i=1}^{k-1} p_i}\left(x - \frac{P_{k-1}}{Q_{k-1}}\right).$$

If we consider the transformation S on $[0, 1)$ defined by $S(x) = T^k(x)$ if

$$x \in \left[\frac{P_{k-1}}{Q_{k-1}}, \frac{P_k}{Q_k}\right),$$

then S is a generalized Lüroth series transformation which was studied in [**BBDK**], and it was shown that S preserves Lebesgue measure, and its natural extension is defined on $[0, 1) \times [0, 1)$ by

$$\mathcal{S}(x, y) = \left(S(x), \frac{P_{k-1}}{Q_{k-1}} + \frac{y}{\beta^{\sum_{i=1}^{k-1} p_i + 1}\alpha^{(k-1)-\sum_{i=1}^{k-1} p_i}}\right) \quad \text{if } x \in \left[\frac{P_{k-1}}{Q_{k-1}}, \frac{P_k}{Q_k}\right).$$

Furthermore, $\mathcal{S}$ preserves the two-dimensional normalized Lebesgue measure and $\mathcal{S}$ is Bernoulli. Consider the projection $\rho : R_0 \times \{0\} \to [0, 1) \times [0, 1)$ given by $\rho(x, y, 0) = (x, y)$. Then, $\rho \circ \mathcal{W} = \mathcal{S} \circ \rho$ and $\mathcal{W}$ and $\mathcal{S}$ are isomorphic, hence $\mathcal{W}$ is Bernoulli. We now use the following theorem to prove that $\mathcal{T}$ is Bernoulli.

THEOREM 4.1. (Saleski's theorem) *Let $(X, \mathcal{B}, \mu, T)$ be a non-atomic Lebesgue space with an automorphism T. Let $A \in \mathcal{B}$ be a subset of X of positive measure and denote by T_A the induced transformation of T on A. Moreover, suppose that we have that T_A is Bernoulli, T is weakly mixing and*

$$H_{\mu_A}\left(\bigvee_{i=1}^{\infty}\bigvee_{j=1}^{\infty} T_A^i Y_j \,\middle|\, \bigvee_{i=0}^{\infty} T_A^i P\right) < \infty,$$

where P is a Bernoulli partition of (A, T_A) and

$$Y_j = \left\{A - \bigcup_{i=1}^{j} T^{-i}A,\ A \cap \bigcup_{i=1}^{j} T^{-i}A\right\}.$$

Then T is a Bernoulli automorphism.

We have the following result.

THEOREM 4.2. *The system $(Z, \bar{\mathcal{B}}, \bar{\lambda}, \mathcal{T})$ is Bernoulli.*

Proof. Note that T is exact, hence mixing, implying that $\mathcal{T}$ is mixing, and therefore weakly mixing; see [**Roh**]. We now apply Saleski's theorem with $A = R_0 \times \{0\}$, $T_A = \mathcal{W}$ and $P = \{R_0^k \times \{0\} \mid k \geq 1\}$ the Bernoulli partition. In our case the sets Y_j are given by

$$Y_j = \left\{\left[\frac{P_j}{Q_j}, 1\right) \times [0, 1) \times \{0\}, \left[0, \frac{P_j}{Q_j}\right) \times [0, 1) \times \{0\}\right\}.$$

Now, the partition P is a refinement of the partition Y_j for all $j \geq 1$, hence $\bigvee_{i=1}^{\infty} \mathcal{W}^i P$ is a refinement of $\bigvee_{i=1}^{\infty}\bigvee_{j=1}^{\infty} \mathcal{W}^i Y_j$ for all $j \geq 1$. This implies that

$$H_{\bar{\lambda}_{R_0\times\{0\}}}\left(\bigvee_{i=1}^{\infty}\bigvee_{j=1}^{\infty} \mathcal{W}^i Y_j \,\middle|\, \bigvee_{i=1}^{\infty} \mathcal{W}^i P\right) = 0,$$

where $\bar{\lambda}_{R_0\times\{0\}}$ denotes the induced measure of $\bar{\lambda}$ on $R_0 \times \{0\}$. Thus, $\mathcal{T}$ is Bernoulli. □

5. *Slow β-expansions*

In this section we consider the case $\alpha = 1/\beta^{\ell}$, for some $\ell \in \mathbb{N}$. In this case

$$T(x) = \begin{cases} \beta x, & x \in \Delta(0), \\ (\beta x - 1)/\beta^{\ell+1}, & x \in \Delta(1). \end{cases}$$

Since $T(1) = (\beta - 1)/\beta^{\ell+1} < 1/\beta^{\ell+1}$, then $T^i\Delta(1) \subset \Delta(0)$ for $i = 1, 2, \ldots, \ell + 1$. That is, whenever $x \in \Delta(1)$, then $T(x), \ldots, T^{\ell+1}(1) \in \Delta(0)$, and $T^{\ell+2}(1) = T_\beta(1)$, where T_β is the greedy transformation, given by

$$T_\beta(x) = \beta x \bmod 1.$$

This implies that T_β is a jump transformation of T, with

$$T_\beta(x) = \begin{cases} T(x), & x \in \Delta(0), \\ T^{\ell+2}(x), & x \in \Delta(1), \end{cases} \quad \text{and} \quad T(x) = \begin{cases} T_\beta(x), & x \in \Delta(0), \\ T_\beta(x)/\beta^{\ell+1}, & x \in \Delta(1). \end{cases}$$

Let $a_n = a_n(x)$ be the nth $(1/\beta^\ell, \beta)$-digit of x as given in (7), and let $d_n = d_n(x)$ be the greedy digits of x, i.e. $d_n = \lfloor \beta T_\beta^{n-1}(x) \rfloor$, $n \geq 1$. From the above we see that whenever $a_n = 1$, then $a_{n+1} = \cdots = a_{n+\ell+1} = 0$. So given the sequence $(a_n(x))_{n\geq 1}$, the sequence $(d_n(x))_{n\geq 1}$ is completely determined; simply remove in $(a_n(x))_{n\geq 1}$ the $\ell + 1$ zeros following every occurrence of 1. Vice versa, knowing $(d_n(x))_{n\geq 1}$, we can construct $(a_n(x))_{n\geq 1}$ by inserting $\ell + 1$ zeros after every occurrence of 1. We formalize this relationship as follows.

Let $s(x) = \inf\{n \geq 1 \mid T^n(x) = T_\beta(x)\}$. Note that

$$s(x) = \begin{cases} 1, & x \in \Delta(0), \\ \ell + 2, & x \in \Delta(1), \end{cases}$$

and we have that $T_\beta(x) = T^{s(x)}(x)$, and if $s(x) = \ell + 2$, then $T(x), T^2(x), \ldots, T^{s(x)-1}(x) \in \Delta(0)$. Set for $i \geq 1$, $s_i(x) = s(T_\beta^{i-1}(x))$, where $s_1(x) = s(x)$. We call s_i the ith *jump time*. Given the $(1/\beta^\ell, \beta)$-digits $(a_n)_{n\geq 1}$ and the greedy digits $(d_n)_{n\geq 1}$ of x, we have

$$a_1 = d_1 \quad \text{and} \quad a_2 = \cdots = a_{s_1} = 0 \text{ if } s_1 = \ell + 2,$$

and for $i \geq 1$,

$$a_{s_1+\cdots+s_i+1} = d_{i+1} \quad \text{and} \quad a_{s_1+\cdots+s_i+2} = \cdots = a_{s_1=\cdots+s_{i+1}} = 0 \text{ if } s_{i+1} = \ell + 2.$$

We now compare 'on a finite level' the $(1/\beta^\ell, \beta)$-expansion of x and its greedy expansion. More precisely, let

$$x = \sum_{n=1}^{\infty} \frac{h_n}{\beta^{\sum_{i=1}^n p_i - \ell(n - \sum_{i=1}^n p_i)}}$$

be the $(1/\beta^\ell, \beta)$-expansion of x, and let

$$x = \sum_{n=1}^{\infty} \frac{d_n}{\beta^n}$$

be its greedy expansion. We have the following result.

THEOREM 5.1. *Let $x \in [0, 1]$ be such, that $T^m(x) \neq 0$ for all $m \geq 0$. Then for any $n \geq 1$ one has*

$$\sum_{m=1}^{s_1+\cdots+s_n} \frac{h_n}{\beta^{\sum_{i=1}^m p_i - \ell(m - \sum_{i=1}^m p_i)}} = \sum_{m=1}^{n} \frac{d_m}{\beta^m}, \tag{11}$$

and

$$\sum_{i=1}^{s_1+\cdots+s_n} p_i - \ell\bigg(\sum_{i=1}^{n} s_i - \sum_{i=1}^{s_1+\cdots+s_n} p_i\bigg) = n. \tag{12}$$

Proof. The proof is done by induction. Let $n = 1$, we have two possible cases.

(i) If $s_1 = s_1(x) = 1$, then $x \in \Delta(0)$, $h_1 = 0$, $p_1 = 1$, and $d_1 = 0$. This implies that both sides of (11) are equal to zero and that both sides of (12) are equal to one.

(ii) If $s_1 = s_1(x) = \ell + 2$, then $x \in \Delta(1)$, $h_1 = 1/\beta^{\ell+1}$, $p_1 = 0$, $d_1 = 1$, $h_2 = \cdots = h_{s_1 = \ell+2} = 0$ and $p_2 = \cdots = p_{s_1} = 1$. Therefore,

$$\sum_{m=1}^{s_1} \frac{h_m}{\beta^{\sum_{i=1}^m p_i - \ell(m - \sum_{i=1}^m p_i)}} = \frac{h_1}{\beta^{p_1 - \ell(1-p_1)}} = \frac{h_1}{\beta^{-\ell}} = \frac{1}{\beta} = \sum_{m=1}^{1} \frac{d_m}{\beta^m},$$

and

$$\sum_{i=1}^{s_1} p_i - \ell\bigg(s_1 - \sum_{i=1}^{m} p_i\bigg) = (s_1 - 1) - \ell = \ell + 1 - \ell = 1,$$

and it follows that (11) and (12) are satisfied.

Assume that the statement holds for $n = k$, we need to show that it holds for $n = k + 1$. Owing to our assumption, we have

$$\sum_{m=1}^{s_1+\cdots+s_k} \frac{h_m}{\beta^{\sum_{i=1}^m p_i - \ell(m - \sum_{i=1}^m p_i)}} = \sum_{m=1}^{k} \frac{d_m}{\beta^m},$$

and

$$\sum_{i=1}^{s_1+\cdots+s_k} p_i - \ell\bigg(\sum_{i=1}^{k} s_k - \sum_{i=1}^{s_1+\cdots+s_k} p_i\bigg) = k.$$

Thus, we only need to show that

$$\sum_{m=s_1+\cdots+s_k+1}^{s_1+\cdots+s_{k+1}} \frac{h_m}{\beta^{\sum_{i=1}^m p_i - \ell(m - \sum_{i=1}^m p_i)}} = \frac{d_{k+1}}{\beta^{k+1}}, \tag{13}$$

and

$$\sum_{i=s_1+\cdots+s_k+1}^{s_1+\cdots+s_{k+1}} p_i - \ell\bigg(s_{k+1} - \sum_{i=s_1+\cdots+s_k+1}^{s_1+\cdots+s_{k+1}} p_i\bigg) = 1. \tag{14}$$

We consider two cases.

(i) If $s_{k+1}(x) = s_1(T_\beta^k(x)) = 1$, then $T_\beta^k(x) = T^{s_1+\cdots+s_k}(x) \in \Delta(0)$,

$$h_{s_1+\cdots+s_k+1}(x) = h_{s_1+\cdots+s_{k+1}}(x) = h_1(T^{s_1+\cdots+s_k}(x)) = 0,$$

$p_{s_1+\cdots+s_{k+1}} = 1$, and $d_{k+1} = 0$. Hence, both sides of (13) are equal to zero. Since

$$p_{s_1+\cdots+s_{k+1}} - \ell(s_{k+1} - p_{s_1+\cdots+s_{k+1}}) = 1,$$

we find that (14) is satisfied.

(ii) If $s_{k+1}(x) = \ell + 2$, then $T^{s_1+\cdots+s_k}(x) = T_\beta^k(x) \in \Delta(1)$, and $T^{s_1+\cdots+s_k+j}(x) \in \Delta(0)$ for $j = 1, \ldots, \ell + 1$. Then,

$$h_{s_1+\cdots+s_k+1} = 1/\beta^{\ell+1}, \quad h_{s_1+\cdots+s_k+2} = \cdots = h_{s_1+\cdots+s_{k+1}} = 0,$$
$$p_{s_1+\cdots+s_k+1} = 0, \quad p_{s_1+\cdots+s_k+2} = \cdots = p_{s_1+\cdots+s_{k+1}} = 1,$$

and $d_{k+1} = 1$. Thus,

$$\sum_{m=s_1+\cdots+s_k+1}^{s_1+\cdots+s_{k+1}} \frac{h_m}{\beta^{\sum_{i=1}^m p_i - \ell(m - \sum_{i=1}^m p_i)}} = \frac{h_{s_1+\cdots+s_k+1}}{\beta^{\sum_{i=1}^{s_1+\cdots+s_k+1} p_i - \ell(s_1+\cdots+s_k+1 - \sum_{i=1}^{s_1+\cdots+s_k+1} p_i)}}.$$

By the induction hypothesis,

$$\sum_{i=1}^{s_1+\cdots+s_k+1} p_i - \ell\left(s_1 + \cdots + s_k + 1 - \sum_{i=1}^{s_1+\cdots+s_k+1} p_i\right) = k,$$

and it follows that

$$\beta^{\sum_{i=1}^{s_1+\cdots+s_k+1} p_i - \ell(s_1+\cdots+s_k+1-\sum_{i=1}^{s_1+\cdots+s_k+1} p_i)} = \beta^{k+p_{s_1+\cdots+s_k+1}-\ell(1-p_{s_1+\cdots+s_k+1})} = \beta^{k-\ell}.$$

We find that

$$\sum_{m=s_1+\cdots+s_k+1}^{s_1+\cdots+s_{k+1}} \frac{h_m}{\beta^{\sum_{i=1}^m p_i - \ell(m-\sum_{i=1}^m p_i)}} = \frac{1}{\beta^{\ell+1}\cdot\beta^{k-\ell}} = \frac{d_{k+1}}{\beta^{k+1}},$$

so (13) holds. Finally,

$$\sum_{i=s_1+\cdots+s_k+1}^{s_1+\cdots+s_{k+1}} p_i - \ell\left(s_{k+1} - \sum_{i=s_1+\cdots+s_k+1}^{s_1+\cdots+s_{k+1}} p_i\right) = (s_{k+1}-1) - \ell = \ell + 1 - \ell = 1,$$

so (14) holds. This proves the theorem. □

6. *(α, β)-expansions in the case $\beta \geq 2$*

As mentioned in the introduction, in the case $\beta \geq 2$, there are two ways to define (α, β)-expansions. In §6.1 a straightforward generalization of the case $1 < \beta < 2$ is considered, while in §6.2 a generalization is discussed which is 'close' to the classical β-expansion. In both cases, the underlying dynamical systems are weakly Bernoulli.

6.1. *Two branches.*

Let $0 < \alpha < 1$ and $\beta \geq 2$, and let the map $T : [0, 1] \to [0, 1]$ be defined as in (3). Using the same notation as in §1.1, we again have for $x \in [0, 1]$, with $T^N(x) \neq 0$ for $N \geq 0$, that

$$x = \sum_{n=1}^{N} \frac{h_n(x)}{\beta^{\sum_{i=1}^n p_i(x)}\alpha^{n-\sum_{i=1}^n p_i(x)}} + \frac{T^N(x)}{\beta^{\sum_{i=1}^N p_i(x)}\alpha^{N-\sum_{i=1}^N p_i(x)}}. \tag{15}$$

We claim that also in this case

$$x = \sum_{n=1}^{\infty} \frac{h_n(x)}{\beta^{\sum_{i=1}^n p_i(x)}\alpha^{n-\sum_{i=1}^n p_i(x)}},$$

cf. (4). Again it suffices to show that

$$\lim_{n\to\infty} \frac{1}{\beta^{\sum_{i=1}^n p_i(x)}\alpha^{n-\sum_{i=1}^n p_i(x)}} = 0,$$

cf. (5). Recall that (5) holds because the series in (6) converges due to Lemma 1.1. This approach does not work for $\beta \geq 2$. Therefore, we give a new proof of (5), which holds for all $\beta > 1$.

For $x \in [0, 1]$, with $T^N(x) \neq 0$ for $N \geq 0$, (15) implies that

$$\sum_{n=1}^{N} \frac{h_n}{\beta^{\sum_{i=1}^{n} p_i(x)} \alpha^{n - \sum_{i=1}^{n} p_i(x)}} < x.$$

We therefore have that

$$0 < S := \sum_{n=1}^{\infty} \frac{h_n(x)}{\beta^{\sum_{i=1}^{n} p_i(x)} \alpha^{n - \sum_{i=1}^{n} p_i(x)}} \leq x. \tag{16}$$

For $k \in \mathbb{N}$, let n_k be defined by

$$n_k = \min\left\{ n \in \mathbb{N};\ \sum_{i=0}^{n-1} 1_{I_1}(T^i(x)) = k \right\},$$

where 1_{I_1} is the indicator function of the set I_1. Since $h_n \neq 0$ for infinitely many $n \geq 1$, the n_k are defined for every $k \in \mathbb{N}$. Note that $n_1 < n_2 < \cdots$ are exactly the 'times' that $h_n \neq 0$. Consequently,

$$S = \frac{\alpha}{\beta} \sum_{k=1}^{\infty} \frac{1}{\beta^{\sum_{i=1}^{n_k} p_i(x)} \alpha^{n_k - \sum_{i=1}^{n_k} p_i(x)}},$$

and it immediately follows from (16) that

$$\lim_{k \to \infty} \frac{1}{\beta^{\sum_{i=1}^{n_k} p_i(x)} \alpha^{n_k - \sum_{i=1}^{n_k} p_i(x)}} = 0. \tag{17}$$

Moreover, for each $k \geq 1$ we have by definition of T that

$$\begin{aligned}
\frac{1}{\beta^{\sum_{i=1}^{n_k} p_i(x)} \alpha^{n_k - \sum_{i=1}^{n_k} p_i(x)}} &> \frac{1}{\beta} \frac{1}{\beta^{\sum_{i=1}^{n_k} p_i(x)} \alpha^{n_k - \sum_{i=1}^{n_k} p_i(x)}} \\
&= \frac{1}{\beta^{\sum_{i=1}^{n_k+1} p_i(x)} \alpha^{n_k + 1 - \sum_{i=1}^{n_k+1} p_i(x)}} \\
&> \frac{1}{\beta} \frac{1}{\beta^{\sum_{i=1}^{n_k+1} p_i(x)} \alpha^{n_k + 1 - \sum_{i=1}^{n_k+1} p_i(x)}} \\
&= \frac{1}{\beta^{\sum_{i=1}^{n_k+2} p_i(x)} \alpha^{n_k + 2 - \sum_{i=1}^{n_k+2} p_i(x)}} \\
&\ \vdots \\
&> \frac{1}{\beta^{\sum_{i=1}^{n_{k+1}-1} p_i(x)} \alpha^{n_{k+1} - 1 - \sum_{i=1}^{n_{k+1}-1} p_i(x)}}.
\end{aligned}$$

By 'sandwiching' we see that the desired result (5) follows from (17), i.e. we have proved the following lemma.

LEMMA 6.1. *Let* $0 < \alpha < 1$ *and* $\beta > 1$, *then*

$$x = \sum_{n=1}^{\infty} \frac{h_n(x)}{\beta^{\sum_{i=1}^{n} p_i(x)} \alpha^{n - \sum_{i=1}^{n} p_i(x)}}.$$

None of the results in §§2, 3, and 4 made use of the fact that $1 < \beta < 2$. Therefore, all of the results from these sections hold for all $0 < \alpha < 1$ and $\beta > 1$. However, the results in §5 depend on the fact that $1 < \beta < 2$; see Lemma 1.1. Note that T is weakly Bernoulli for $\beta > 2$ follows as well from [**E**].

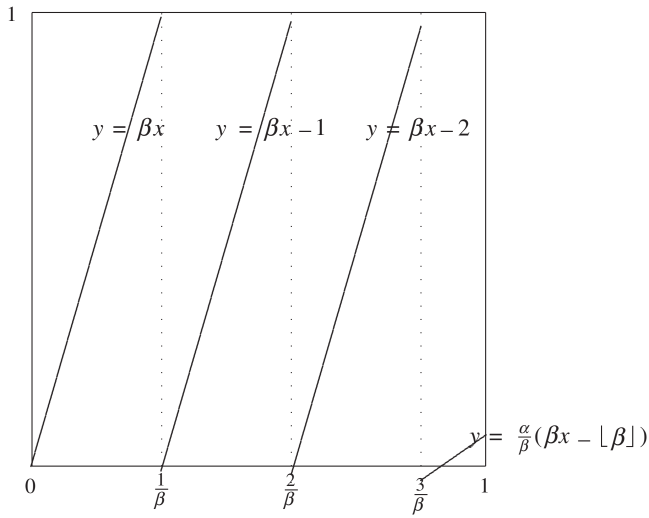

FIGURE 2. The map T with more than two branches; here $\alpha = 0.71$ and $\beta = 3.5$.

6.2. *More than two branches.* Let $0 < \alpha < 1$ and $\beta \geq 2$. As a variation on (1), the map $T : [0, 1] \to [0, 1]$ is defined by

$$T(x) = \begin{cases} \beta x \pmod 1), & x \in [0, \lfloor\beta\rfloor/\beta), \\ \dfrac{\alpha}{\beta}(\beta x - \lfloor\beta\rfloor), & x \in [\lfloor\beta\rfloor/\beta, 1]; \end{cases} \tag{18}$$

see also Figure 2.

To obtain expansions, we need to rewrite T as in §1.1; for $x \in [0, 1]$, set

$$p = p(x) = \begin{cases} 1, & x \in I_i, \, i = 0, \ldots, \lfloor\beta\rfloor - 1, \\ 0, & x \in I_{\lfloor\beta\rfloor}, \end{cases}$$

and

$$h = h(x) = \begin{cases} i, & x \in I_i, \, i = 0, \ldots, \lfloor\beta\rfloor - 1, \\ \dfrac{\alpha}{\beta}, & x \in I_{\lfloor\beta\rfloor}, \end{cases}$$

where $I_i = [i/\beta, (i+1)/\beta)$, for $i = 0, 1, \ldots, \lfloor\beta\rfloor - 1$ and $I_{\lfloor\beta\rfloor} = [\lfloor\beta\rfloor/\beta, 1]$. Then $T(x) = \beta^{p(x)}\alpha^{1-p(x)}x - h(x)$. For $n \geq 1$, define $p_n(x) = p(T^{n-1}(x))$ and $h_n(x) = h(T^{n-1}(x))$. Then, if $T^N(x) \neq 0$, we have

$$x = \sum_{n=1}^{N} \frac{h_n(x)}{\beta^{\sum_{i=1}^n p_i(x)}\alpha^{n-\sum_{i=1}^n p_i(x)}} + \frac{T^N(x)}{\beta^{\sum_{i=1}^N p_i(x)}\alpha^{N-\sum_{i=1}^N p_i(x)}}.$$

Thus, we see that if for some m, $T^m(x) = 0$, and m is the least positive integer with this property, then x has a finite expansion of the form

$$x = \sum_{n=1}^{m} \frac{h_n(x)}{\beta^{\sum_{i=1}^n p_i(x)}\alpha^{n-\sum_{i=1}^n p_i(x)}}.$$

Suppose now that $T^n(x) \neq 0$ for all $n \geq 1$. We claim that in this case

$$x = \sum_{n=1}^{\infty} \frac{h_n(x)}{\beta^{\sum_{i=1}^{n} p_i(x)} \alpha^{n - \sum_{i=1}^{n} p_i(x)}}.$$

As in §6.1, this follows from Lemma 6.1 (note that in the proof of Lemma 6.1 we did not use the fact that T has two branches).

In fact, we have a stronger result; not only do we have that

$$\lim_{n \to \infty} \frac{1}{\beta^{\sum_{i=1}^{n} p_i(x)} \alpha^{n - \sum_{i=1}^{n} p_i(x)}} = 0,$$

but even that

$$\sum_{n=1}^{\infty} \frac{1}{\beta^{\sum_{i=1}^{n} p_i(x)} \alpha^{n - \sum_{i=1}^{n} p_i(x)}} < \infty.$$

This follows from the following lemma, which is a straightforward generalization of Lemma 1.1.

LEMMA 6.2. *Let* $0 < \alpha < 1$ *and* $\beta > 1$, *and let the map* T *be defined as in (18). Then we have that:*

(i) $T(I_{\lfloor \beta \rfloor}) \subset I_0$;

(ii) *let* $k = k(\alpha)$ *be the unique non-negative integer for which* $(1/(\beta^{k+1})) < \alpha \leq 1/\beta^k$, *then* $T^i(I_{\lfloor \beta \rfloor}) \subset I_0$ *for* $1 \leq i \leq k+1$.

A proof similar to that in Proposition 1.1 gives the desired result.

The results from §§2, 3, 4 and 5 can be extended to the present case by making slight adjustments to the proofs.

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