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Learning trajectory for equivalent fraction learning: An insight

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Abstract

Equivalent fraction is a sub-topic of fractions that highly contributes to explaining the basic concepts of fractions. However, this topic is one of the most challenging topics for students as it involves an advanced and formal concept and various representations. This study aims to present the preliminary result of the learning trajectory on equivalent fractions. This design research consisted of three stages: preparation, implementation, and retrospective analysis. The learning trajectory in the form of a hypothetical learning trajectory (HLT) was designed in the context of measuring cups. The data was collected through documentation, interviews, and class observations. The HLT was implemented to investigate students' actual learning trajectories. The findings showed that the measuring cup context assists students to easily perceive those different fractions may have the same quantity (equivalent fractions). The learning trajectory consists of two activities. The first one aims to introduce a measurement concept of fractions to students. The second one aims to help students construct the concept of equivalent fractions. Finally, the study findings contribute to further development of learning trajectory on equivalent fractions.

Keywords: Design Research, Equivalent Fractions, HLT, Measurement

Introduction

One of the mathematics topics in elementary schools designed by the Indonesian Ministry of Education and Culture is fractions. Fraction is classified as a topic with a formal concept and an advanced level (Wilkins & Norton, 2018). It is also a mathematical concept with diverse representations (Pedersen & Bjerre, 2021). These factors cause fractions a complicated topic for students (Siegler et al., 2013). One of the sub-topics in fractions considered complex is equivalent fractions (Pedersen & Bjerre, 2021). Whereas the equivalent fraction is a basic fraction concept that contributes to the addition and subtraction of fractions, approaches to the division of fractions, and the relationship between fractions and decimals (Kara et al., 2018).

Besides, equivalent fraction also contributes to the gap between students with mathematics learning difficulties and those with excellent achievements (Hacker et al., 2019).

Students' knowledge of the natural-numbers concept representing a quantity may lead to bias towards the fraction concept (Stafylidou & Vosniadou, 2004; Ni & Zhou, 2005) and equivalent fractions, which means that more than one fraction may have the same quantity (Wilkins & Norton, 2018). To illustrate, students have difficulty determining which fraction has a greater value, 1/4 or 1/2. As in the concept of the natural numbers, students understand that 4 is greater than 2. This condition drives students to understand equivalent fractions only as a procedure of multiplying or dividing the denominator and numerator by the same number (Wong, 2010). According to Pedersen and Bjerre (2021), to understand equivalent fractions, students can compare several different fractions, but the fractions refer to the same relative amount.

There is an approach in mathematics learning that focuses on students' interactivity and contributions in learning, namely Realistic Mathematics Education of Indonesia or known as *Pendekatan Matematika Realistik Indonesia* (PMRI) (Putri, 2011; Zulkardi & Putri, 2010; Zulkardi, 2002). PMRI is an adaptation of Realistic Mathematics Education (RME) to the Indonesian context. The philosophy of RME, according to Freudenthal, the inventor of RME, is that mathematics serves as a human activity, which means providing students the opportunity to discover mathematics by themselves (Zulkardi, 2002).

When designing learning, the teacher needs to understand students' thinking and generate assumptions or hypotheses of the possible responses given by students. Thus, the learning presented to students is tailored to their needs (Wijaya et al., 2021). The learning design that considers students' abilities is called the Hypothetical Learning Trajectory (HLT). HLT means understanding students' ways of thinking and developing the learning based on conjectures of students' thinking (Wijaya et al., 2021). An HLT consists of learning objectives, activities, and conjectures of students' thinking in each learning activity (Simon, 1995).

Therefore, this study aims to design a learning trajectory or an HLT by providing students an opportunity to experience comparing different fractions but referring them to the same amount by measuring water using a measuring cup. Problems presented in the HLT are also expected to bridge students to have a thorough understanding of equivalent-fraction concepts (Streefland, 1991; Haris & Putri, 2011).

Methods

The current study employed a design research method of validation study type to design an intervention that aims to support students' understanding of the equivalent-fraction concept. This study comprised three stages: preparation for an experiment, implementation, and retrospective analysis (Putri et al., 2021; Putri & Zulkardi, 2017; Bakker, 2004). The participants were three students from Class IV A and 11 students from Class IV B at the Elementary School of IBA Palembang.

The first stage of this research was preparing for an experiment by conducting literature reviews and interviewing the fourth-grade teachers at the Elementary School of IBA Palembang. This literature review serves as a theoretical basis for equivalent fractions and equivalent fraction learning, which will be fundamental to designing an initial HLT. Meanwhile, the interview aims to identify the initial abilities of students in Class IV B at the Elementary School of IBA Palembang. The researchers and the teachers adjusted the HLT to the conjecture of students' thinking and designed learning activities and evaluations appropriate for students. The HLT design that had been developed is depicted in Table 1.

Table 1. Design of HLT

Aims	Activity	Conjecture of Students' Thinking
To understand that the fraction $\frac{m}{n}$ means the fraction $\frac{1}{n}$ is repeated m times	Pouring the water into a $\frac{1}{n}$ cup	Students can iterate as expected and are able to conclude that the fraction m/n means the fraction 1/n with m iterations.
To understand the concept of equivalent fractions	Pouring the water into $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ and $\frac{1}{16}$ cups	• Good students are able to carry out the iteration process and find the fact that the measurement result of the second quantity is equivalent to the ½ cup. • High-ability students may conclude that the equivalent fraction of a fraction a/b can be determined by multiplying to the fraction with the value of 1 or $\frac{c}{c}$, thereby not changing its value. • Moderate-ability students may conclude that to obtain a fraction equal to $\frac{a}{b}$, they need to multiply a by c then multiply b by c. • Low-ability students may merely conclude that the results of the second iteration are equivalent to $\frac{1}{2}$, but they cannot generalize their reason.

The experiment implementation stage consisted of two cycles. In the first cycle, a pilot experiment was conducted to identify the feasibility of the initial HLT design and collect the data needed to revise the HLT. In the second cycle, a teaching experiment was carried out to gather the necessary data for this study, specifically on how the improved HLT helps students learn equivalent fractions. The data was obtained from students' worksheets, evaluation questions, and observation results during the learning process.

The final stage is a retrospective analysis, in which the researchers compared the HLT to the actual learning process in the teaching experiment. The analysis includes possible causes and improvements that can be made in the future, as design research is not intended to be successful but to reveal how and why this intervention works (Doorman, 2005). Figure 1 presents a flowchart of this research design.

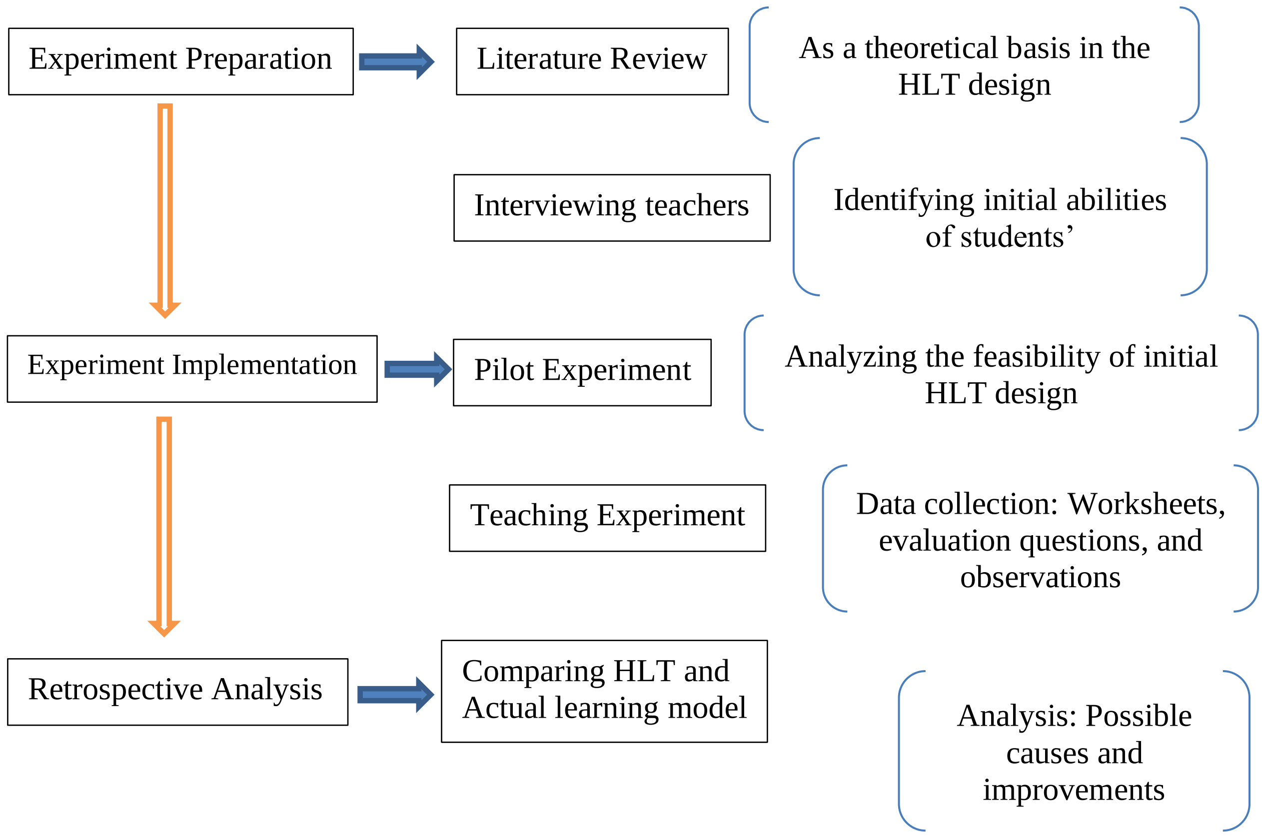


Figure 1. Research flow

Hypothetical Learning Trajectory (HLT)

Before designing the HLT, the researchers conducted a literature review to formulate the initial design of learning objectives and activities. The learning objectives are primarily referred to the mathematics education curriculum in Indonesia. Based on Streefland's (1991) theory–focusing on providing students with concrete materials, including context models–the researchers arranged learning activities that focus on measuring the amount of water to provide students experiences of representing fractions. Streefland (1991) also states that the distribution process can produce equivalent fractions, which is fundamental for students at the symbolic level.

Results and Discussion

In this study, we designed a learning trajectory on equivalent fractions for fourth-grade students at elementary school, which contained two activities. The activities include pouring the water into the $1/n$ cup and pouring the water into the 1/2, 2/4, 4/8, and 8/16 cups. The stage of implementing the experiment comprises a pilot experiment and a teaching experiment. Each activity will be described in the following section. To solve the problems on the worksheet, students were provided measuring cups with the size of 1/16, 1/8, 1/4, 1/2, 3/4, and 4/4 and water.

Activity 1: Introduction of measurement

In the first activity, students poured the water into a 4/4 cup until it is full using a 1/4 cup (Figure 2). The purpose of this activity is to introduce a measurement concept of fractions to students by measuring the number of pouring until the 4/4 cup is full. This activity also introduces other examples of measurement to students.

Figure 2. Students poured the water

Students attained different answers from Activity 1. Some students answered nine times of pouring until the 4/4 cup was full. Some responded seven times of pouring, and others stated four times of pouring. In this case, students had not yet understood that the 1/4 cup is a fractional form of a 4/4 cup partitioned into four equal parts. The teacher posed a question to all students as to why they answered the question differently.

Teacher : *"…. Who wants to give an argument of why some answered 9, 7, and 4?"*
Student Q : *"Because we did not fill up the ¼ cup."*
Teacher : *"If you fill up the ¼ cup, what is the result?"*
Student A : *"4"*

Student Q initially answered nine, and Student A answered four. To reassure those students understand the rules of Activity 1, they are required to work on the second problem, to wit filling up the 3/4 cup using a 1/4 cup. All students answered three to this problem; then, the teacher asked questions that led students to the conclusion.

Teacher : *"How many times of pouring do we need to fill up the 4/4 cup?"*
Student S : *"4 times"*
Teacher : *"How many times of pouring do we need to fill up the 3/4 cup?"*
Student : *"3 times"*
Teacher : *"If we have a big cup with a size of m/4 and we need to fill up the cup with water using a ¼ cup, how many times do we need to pour the water?"*
Student A : *"m times."*
Teacher : *"So, where does the number come from?"*
Student : *"The numerator."*

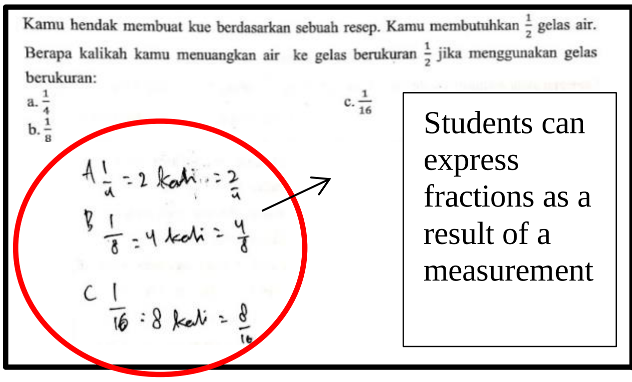
Based on Activity 1, students performed an understanding of fraction subconstructs as a measurement concept (Wilkins & Norton, 2018), and this aligns with the conjectures proposed by the teachers and researchers in the HLT.

Activity 2: Investigation students' understanding of the measurement concept

After successfully constructing a fraction as a measurement concept, we continued to Activity 2 using a measuring cup with a denominator other than 4. In this activity, students were requested to determine the number of times they needed to pour the water to fill up the ½ cup using the 1/4, 1/8, and 1/16 cups (Figure 3). This activity aims to investigate students' understanding of the measurement concept of fractions and construct the concept of equivalent fractions.

Figure 3. Student activity to investigate the measurement concept of fractions

Figure 4 shows that the students' answers on the worksheet reveal that they understand the measurement concept of fractions through the iterations of pouring water using cups of different sizes (1/4, 1/8, and 1/16).

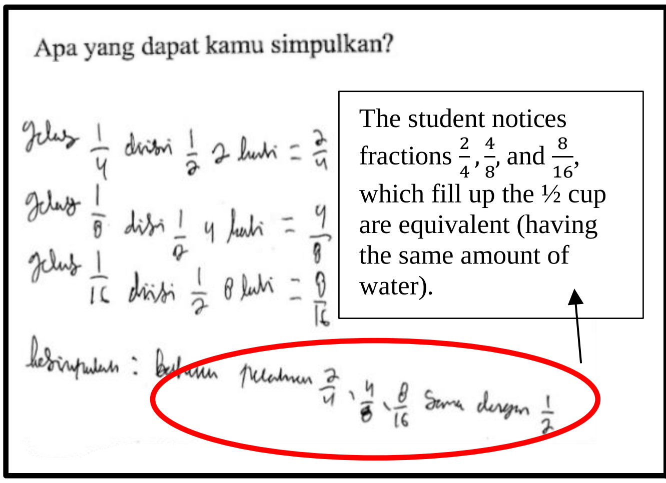


You want to make a cake based on a recipe. You need $a\,\frac{1}{2}$ cup of water. How many times do you have to pour the water into a $\frac{1}{2}$ cup if you use cups with the following sizes:

a. $\frac{1}{4}$
b. $\frac{1}{8}$
c. $\frac{1}{16}$

Figure 4. Student's answer about the measurement concept of fractions through the iterations of pouring water

Furthermore, the point of HLT is drawing conclusions from Activity 2, which aims to construct students' understanding of equivalent fractions (Figure 5). The teacher asked students to analyze what sizes of cups they used. Various sizes of cups (in fractions) orient students to comprehend those three different fractions may have the same value (the same amount of water) (Pedersen & Bjerre, 2021).



What can you conclude from this activity?

Figure 5. Student's answer in drawing conclusion from Activity 2

The following is an excerpt between the teacher and Student T, a low-ability student. Student T understood that the amount of water is equal but had not yet mentioned it in a notation of equivalent fractions. This student's response agrees with the conjecture listed in the designed HLT.

Teacher : *"...what size of cup does the amount of water represented by fractions* $\frac{2}{4}, \frac{4}{8}$, *and* $\frac{8}{16}$ *fill up?"*

Student T : *"A cup with a size of* $\frac{1}{2}$*"*

Teacher : *"Then, what can you conclude?"*

Student T : *"The amount of water is the same"*

Teacher : *"Okay, what does it mean by the amount of water is the same?"*

The teacher then gave an opportunity to Student J, who was in the same group as Student T, to assist Student T.

Student J : *"The amount of water from the* $\frac{2}{4}, \frac{4}{8}$, *and* $\frac{8}{16}$ *cups are equal to the amount of water in the* $\frac{1}{2}$ *cup"*

Teacher : *"Could you please write it down, how to write equal?"*

Student T : *(Writing symbol =)*

Teacher : *"Student J, could you please mention which one is equal?"*

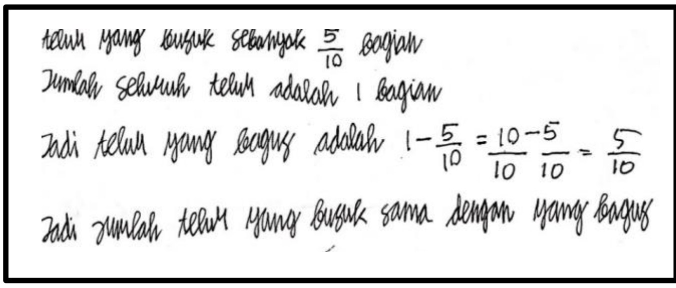
Student J : *"*$\frac{2}{4}, \frac{4}{8}, \frac{8}{16}$ *is equal to* $\frac{1}{2}$*"*

Teacher : *"Could you please write it down?"*

Student T : *(Writing* $\frac{2}{4}, \frac{4}{8}, \frac{8}{16} = \frac{1}{2}$*)*

The teacher also clarified that each of the fractions $\frac{2}{4}, \frac{4}{8}$, and $\frac{8}{16}$ is equivalent to $\frac{1}{2}$. Based on Activity 2, low-ability students were able to notice more than one fraction equivalent to $\frac{1}{2}$. This finding accords with Pedersen and Bjerre (2021), stating that when the whole (denominator value) varies but all represent an equivalent quantity, students understand that the point is proportionality (equality).

To evaluate students' understanding of equivalent fractions, the teacher passed out evaluation questions at easy, moderate, and difficult levels. The following figures present the answers of three students with various abilities to Question 2 (moderate level), namely a high-ability student (Figure 6), moderate-ability student (Figure 7), and low-ability student (Figure 8).



The rotten eggs are $\frac{5}{10}$ parts. The number of all eggs is 1 part. So, the good eggs are $1 - \frac{5}{10} = \frac{10}{10} - \frac{5}{10} = \frac{5}{10}$. Thus, the number of rotten eggs is the same as the number of good eggs.

Figure 6. The answer of a high-ability student

For the medium-level question, it appears that students used various strategies. Figure 6 shows that the high-ability student mastered operations of fractions; thus, he was able to notice that the eggs with the good condition have a fraction equivalent to the rotten ones.

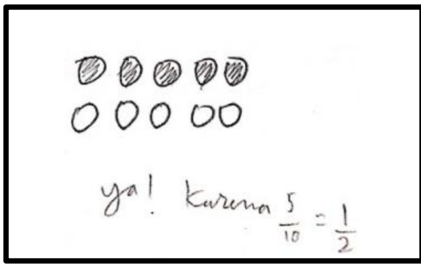
	Yes! Because $\frac{5}{10} = \frac{1}{2}$

Figure 7. The answer of a moderate-ability student

Furthermore, Figure 7 shows that the moderate-ability student illustrated five rotten eggs out of ten eggs in total. He then concluded from the picture that the rotten eggs are half, and the remaining half is in good condition.

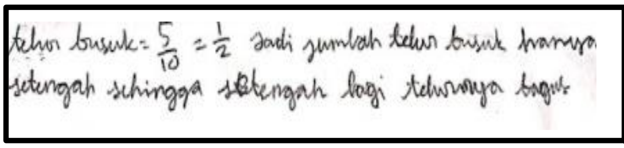
	The rotten eggs= $\frac{5}{10} = \frac{1}{2}$. So, the number of rotten eggs is only a half. Another half is in good condition.

Figure 8. The answer of a low-ability student

Meanwhile, the low-ability student answered 5/10=1/2 without explaining why 5/10 equals 1/2. In brief, based on the students' responses, they were able to notice 5/10 is a half as presented in Figure 8.

This diversity of strategies is also evident in the students' answers to complex questions. Any solutions given by the students have a similar idea that 18/24 is equal to 6/8 and is also equal to 3/4. Therefore, most students demonstrated their understanding of equivalent fractions as the distribution result (Streefland, 1991). However, only one student did provide an inaccurate solution, as shown in Figure 9.

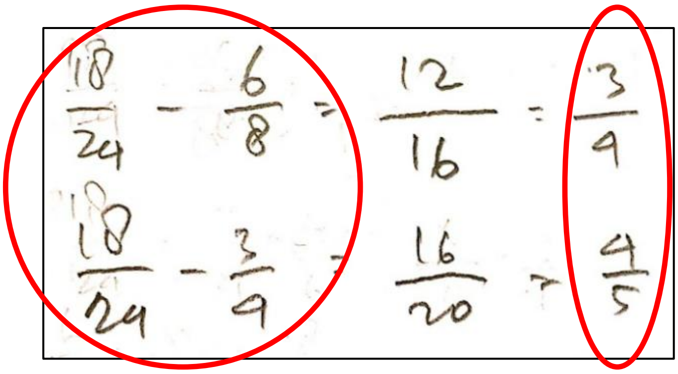


Figure 9. Student's answer with an inaccurate solution

The student did not understand that 18 cakes distributed to 24 students will have the same result as 6 cakes distributed to 8 students, or 3 cakes distributed to 4 students. Instead, he looked at the equivalence of the differences between the initial fraction 18/24 and the given distribution of fractions, 6/8 and 3/4. Table 2 depicts the comparison of the researchers' conjectures toward students' thinking and the actual evaluation activities.

Table 2. HLT as a result of retrospective analysis

Objective	Conjecture of Students' Thinking	Students' Thinking in the Actual Classroom
Through exercises, students are expected to apply knowledge and understanding of equivalent fractions.	**Question 1** • High, moderate, and low-ability students can determine two fractions equivalent to $\frac{1}{3}$	**Question 1** • High, moderate, and low-ability students can determine two fractions equivalent to $\frac{1}{3}$
	Question 2 • High, and moderate-ability students likely understand $\frac{5}{10}$ is equivalent to $\frac{1}{2}$ • Low-ability students may need visualizations (pictures) to notice that $\frac{5}{10}$ is equivalent to $\frac{1}{2}$	**Question 2** • High, moderate, and low-ability students understand that the number of rotten eggs is the same as the good ones by noticing that $\frac{5}{10}$ is equivalent to $\frac{1}{2}$
	Question 3 • High and moderate-ability students are likely to construct fractions $\frac{6}{8}$ and $\frac{3}{4}$, then determine the two fractions are equivalent. • Low-ability students are likely not able to construct fractions $\frac{6}{8}$ and $\frac{3}{4}$.	**Question 3** • High and moderate-ability students can construct fractions $\frac{6}{8}$ and $\frac{3}{4}$, and determine those are equivalent. • Low-ability students give an irrelevant answer.

Table 2 shows that low-ability students have not been able to construct fractions as a measurement concept. The students' view on different fractions with equivalent quantities is essential to construct their understanding of equivalent fractions (Pedersen & Bjerre, 2021).

The final design and the results of this learning trajectory implementation contribute to generating several activities that support students in constructing their understanding of equivalent fractions. Table 3 presents student evaluation test results regarding equivalent fractions. In general, students have reached the minimum achievement standard. Therefore, the designed learning trajectory can be employed as an alternative for equivalent fraction learning.

Table 3. Students' achievement

The Number of Students	
Pass (satisfactory)	Fail (unsatisfactory)
10	1

The results of the HLT implementation reveal that the design of the learning trajectory based on the PMRI philosophy provides opportunities for students to find the concept of equivalent fractions. It can be employed as a learning activity that supports students in constructing their understanding of equivalent fractions. From Activity 1, students understand the measurement concepts of fractions through iteration activities of pouring water using a measuring cup. Subsequently, Activity 2 demonstrates visualizations of different fractions that have the same quantity (equivalent) (Pedersen & Bjerre, 2021).

Furthermore, the evaluation results indicate that students can understand the concept of equivalent fractions and notice the equivalence of two fractions without utilizing a measuring cup. Based on the evaluation results on Question 3, one would express those students comprehend fractions as distributing objects (Streefland, 1991) and recognize the equivalence of two different distributions.

As a concluding remark, the results of the experimental design reveal that the designed activities can stimulate students' understanding of equivalent fractions. All students' answers from the beginning of Activity 1 and classroom discussion depict how students construct their understanding of fractions as a measurement concept. It is followed by Activity 2, which builds students' understanding of equivalent fractions. Therefore, this study contributes to producing a mathematics learning design on equivalent fractions by directly engaging students in the activities.

Conclusion

This research finding is a learning trajectory for students in constructing the concept of equivalent fractions using concrete objects. The activities focus on constructing the measurement concept of fractions and exploring various fractions that have the same quantity. This important point is attained when students are able to conclude that the quantity of the fractions 2/4, 4/8, and 8/16 is equal to the quantity of fraction 1/2.

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Conflicts of Interest

No conflict of interest is reported.

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