Chapter 13
Developing and Applying the Law of Cosines:
Using Star Maps as a Context

Vecihi S. Zambak
Monmouth University, USA

Budi Mulyono
Sriwijaya University, Indonesia

EXECUTIVE SUMMARY

In history, geometry was founded more as a practical endeavor than a theoretical one. Early developments of the branch portray philosophers' attempts to make sense of their surroundings, including the measurement of distances on earth and in space. Such a link between earth and space sciences and geometry motivated us to develop and implement a multidisciplinary lesson focusing on the conceptual understanding of the law of cosines in the context of astronomy. In our content specific STEAM lesson, the authors aimed to facilitate an understanding of the law of cosines in ninth grade students, and then apply the law in a star map task to find approximate distances between stars. The second part of the lesson also included the use of an instructional technology to support students' work with the star map task. In the conclusion, the authors discuss possible ways to improve the quality of their STEAM education efforts for the given context.

DOI: 10.4018/978-1-5225-9631-8.ch013

INTRODUCTION

As research indicates, students experience significant gains and achievement from interdisciplinary instructional approaches in developing mathematics and science content knowledge; as a result, STE(A)M Education has globally become the ultimate goal of educational programs (Bybee, 2010; Marginson, Tytler, Freeman, & Roberts, 2013; Morrison & McDuffie, 2009; Venville, Rennie, & Wallace, 2004). With an emphasis on creativity and artistic processes, STEAM Education is now defined as the intentional integration of concepts, practices, and design features of science, technology, engineering, art, and mathematics subjects (Bequette & Bequette, 2012; Gess, 2017). Yakman (2010) discusses how the application of STEAM Education, in the context of a classroom, occurs in five different levels: 1) Universal level, 2) Integrated level, 3) Multidisciplinary level, 4) Discipline Specific level, and 5) Content Specific level. In this chapter we define *Content Specific STEAM Education* as the detailed study of a content area with the integration of two or more STEAM disciplines, one of which is the main focus. In the first part of our lesson, ninth grade students were guided to collaboratively discover *the law of cosines*. Then they solved a mathematics problem dealing with *Earth and space sciences*. In the second part of our lesson, students used an applet built in a dynamic geometry environment. In other words, we aimed to integrate the disciplines of mathematics, science, and technology using *discovery learning* as the main instructional strategy.

The law of cosines can be called the extension of the Pythagorean Theorem for acute and obtuse triangles. Even though the concept of cosine was not developed during the Euclid's time, the Elements includes the same equation with different notations and representations. The law makes it possible to find an unknown angle or side when two sides and the angle between them are known or when three sides are known.

Since the law of cosines includes several notations and concepts, many learners find it difficult to retrieve it from their memory and use it when it is needed to solve a problem. There are strategies to reduce the learner's memory load for the distance formula and the law of cosines. For example, McMullin (2003) used computer algebra systems to reduce students' memory load while they were solving a mathematical problem using the law of cosines. Whereas instructional technology usage in his article might have enabled students to find the correct answer for the problem quite promptly, McMullin's approach does not guarantee students' conceptual understanding of the law. To be able to conceptually understand the law of cosines, we believe that students need to synthesize several connected terms and notions in trigonometry and geometry.

Another way to reduce memory load is to use discovery learning approaches that enable learners to analyze and synthesize laws from triangles through deductive reasoning. Discovery learning has varying meanings, from the use of open-ended student projects to inquiry oriented activities. In general terms, it can be defined as a curriculum structure through which students are guided by questions to discover targeted concepts, terms and notions (Hammer, 1997). This approach was found to be more beneficial for learners who had a familiarity with the concepts and facts covered during the instruction (Tuovinen & Sweller, 1999).

CONTENT INFORMATION

The lesson we present in this chapter aims to address many content and practice standards active in the U.S. education system. The following Common Core State Standards for Mathematical Practice have been targeted within the lesson: MP1. Make sense of problems and persevere in solving them; MP2. Reason abstractly and quantitatively; MP4. Model with mathematics; MP5. Use appropriate tools strategically; and MP7. Look for and make use of structure (Common Core State Standards Initiative [CCSSI], 2010). In addition, two Common Core Content Standards shaped the mathematical tasks for our lesson:

High School: Geometry » Similarity, Right Triangles, & Trigonometry » Apply trigonometry to general triangles:

10. (+) Prove the Laws of Sines and Cosines and use them to solve problems.

11. (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces). (CCSSI, 2010)

Our lesson is also aligned with the *International Society for Technology in Education (ISTE) Standards for Students*. More specifically, the second part of our lesson and the use of the applet aimed to enable students to become Empowered Learner(s), Knowledge Constructor(s), and Computational Thinker(s) (ISTE, 2016). The following specific ISTE Standards for Students were central to our lesson:

1c. Students use technology to seek feedback that informs and improves their practice and to demonstrate their learning in a variety of ways.

3d. Students build knowledge by actively exploring real-world issues and problems, developing ideas and theories and pursuing answers and solutions.

5b. Students collect data or identify relevant data sets, use digital tools to analyze them, and represent data in various ways to facilitate problem-solving and decision-making. (ISTE, 2016)

Finally, the "Universe and its Stars" was tangentially used as the Disciplinary Core Idea(s) of Earth and Space Sciences adapted from the *Next Generation Science Standards* (Next Generation Science Standards Lead States, 2013). That being said, it is worth stating that mathematical content and practice standards were the primary focus of our lesson; technology and science standards were secondarily integrated into the instruction. (To make sense of our rationale for the intentional centralization of mathematics in our lesson planning and implementation, please see the definition of Content Specific STEAM Education in the Introduction section of this chapter.)

CASE DESCRIPTION

The Goals and Settings

We established two main goals for the development and implementation of our interdisciplinary lesson: 1) effectively integrate the content of Earth and space sciences and Information Communication Technologies (ICT) in a middle school setting, and 2) implement a discovery learning approach as the leading instructional method. To do that, we developed our lesson plans around student-centered tasks. The authors collaboratively designed the lesson plans, interdisciplinary tasks, and a conducive applet.

We applied our lesson plans in an international school located in Hilversum, the Netherlands. The cooperating teacher behaved as an external advisor for the development and revisions of the lesson plans providing support for possible technical issues that may have emerged during the implementation of lesson plans. The school environment and technological infrastructure also enabled us to use ICT during the instruction. Every student had their own personal laptop.

After negotiating with the cooperating teacher, we selected the law of cosines as the major content focus of our lesson. We considered the logical construction and conceptual understanding of the law of cosines as precursors to student application of the law. For this reason, we developed the first part of our lesson, *My Theory* task. As we mentioned before, Earth and space sciences were chosen as the theme to apply and integrate the law of cosines to the science content. This theme allowed us to develop the second part of our lesson, *Star Map* task. These two tasks were implemented during two class meetings (i.e., a total of 90 minutes) in a ninth-grade class, including seventeen students.

Lesson Plan

Learning Objectives

The following three learning objectives and goals were set for our lesson:
The students will be able to…

1. Construct the law of cosines using trigonometric ratios and the Pythagorean Theorem within various triangles (i.e., obtuse or acute triangles).
2. Identify the characteristics of triangles requiring the use of the law of cosines to find unknowns in problem situations.
3. Solve realistic and contextual problems by applying the law of cosines and using ICT to collect and make sense of data.

Materials

My Theory Task:

- PowerPoint Slides (*My Theory*)
- Teacher PC Station
- Projector
- Student Worksheet Part 1
- *My Theory* Solution Sheet[1]

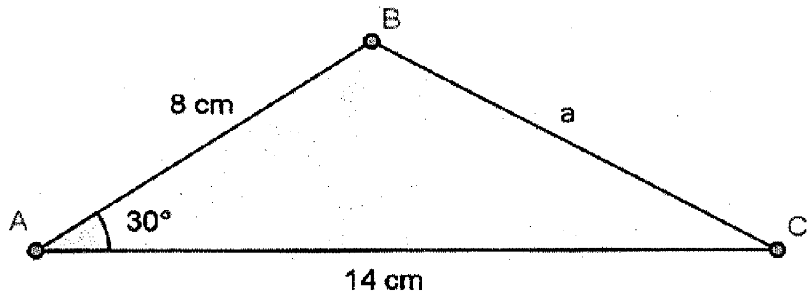
Star Map Task:

- PowerPoint Slides (Star Map)
- Teacher PC Station
- Projector
- Student Worksheet Part 2
- Star Map Extension Task
- PC/Laptops
- Calculator

Tasks

For the first part of our lesson, *My Theory* task, we aimed to give students an opportunity to construct the law of cosines by themselves instead of directing them to memorize without conceptually understanding the content). The *My Theory* task included two parallel problems (see Figure 1 and 2). While *My Theory* Problem 1

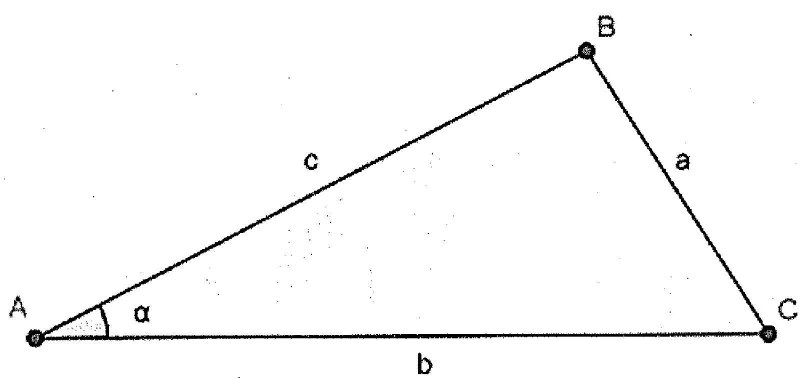
Figure 1. My Theory Problem 1



involved the lengths of two line segments and the angle between them, *My Theory* Problem 2 was the abstraction of the first problem which required students to use algebraic reasoning and manipulations. To be able to solve the second problem, the students needed to recall the trigonometric ratios and the geometrical theorems they used while solving the first problem, and then convert them into related letters.

The Student Worksheet Part 1 also includes hints and guiding questions to support and scaffold students' problem solving processes. For example, in *My Theory* Problem 1, students were first asked to find the height of the triangle drawn perpendicular to the |*AC*| line segment, which would allow them to apply their previous knowledge about trigonometric ratios and the Pythagorean Theorem. This would also make it possible for the students to find the unknown length *a* from the givens. *My Theory* Problem 2 was specifically designed for students to write down the law of cosines

Figure 2. My Theory Problem 2



according to their conceptual understanding, based on their solutions for the first problem given.

My Theory task was built as a small-group activity, and three different versions of the task were therefore created. Versions differed according to the shape of the triangle in *My Theory* Problem 1. While the triangle was acute in two versions, it was obtuse in the third version (see Figure 3).

For the second part of our lesson, the *Star Map* task, we aimed to present a real-world context connected with the history of the subject. The Student Worksheet Part 2 represented a 2D star map including some stars and the Earth at the center (see Figure 4). A table informing the actual distances of the stars in the star map to the Earth was additionally provided (see Table 1). To be able to solve the main problem in the Star Map task, students were asked to use the given information on the star map, the table, and an applet.

Students were expected to find the distance between two stars (e.g., Alpha Centauri and Barnard) by applying the law of cosines. To do that, students needed to know distances of each star from the Earth, which were given in the table, and also the angle between the line segments from each star to the Earth. The angle was not given in the table; students were asked to use the applet to find the missing information in the question.

Technology Integration

The applet was designed to find the angle between the line segments from each star to the Earth by playing with three points that should be placed on two stars and the Earth (see Figure 5 for screenshot). The applet was designed with GeoGebra, a dynamic geometry software. The applet was delivered to each student's laptop through a USB stick. The Student Worksheet Part 2 explicitly guided students to play with the applet on their laptops before beginning to work on the problem. We did not provide students with any additional instruction explaining how to use the applet.

Figure 3. Three different versions of My Theory Problem 1

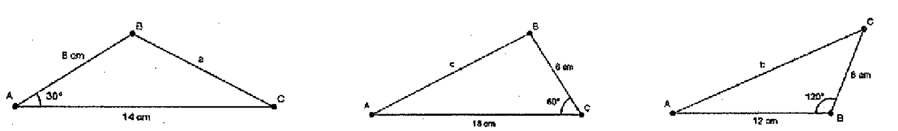
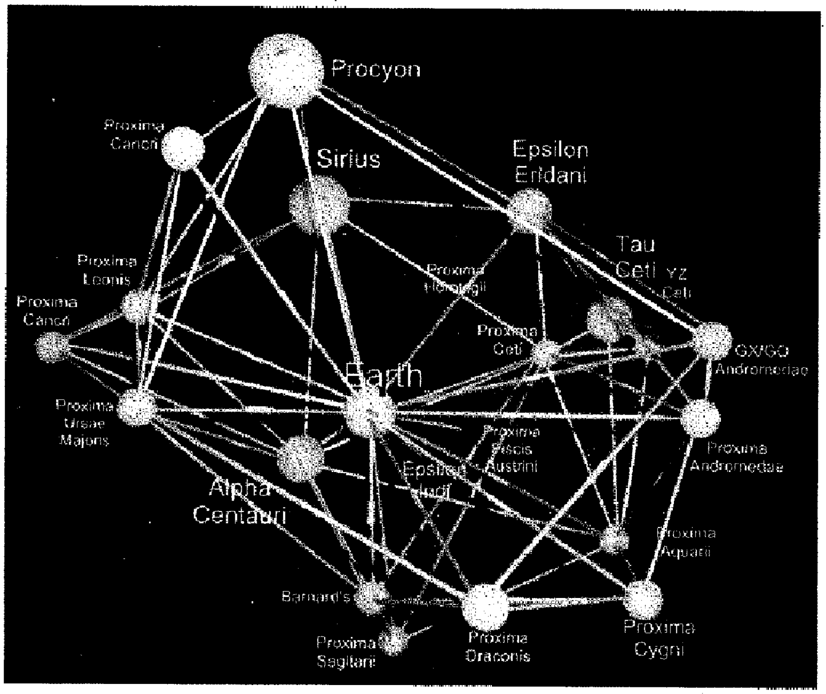


Figure 4. 2D Star Map



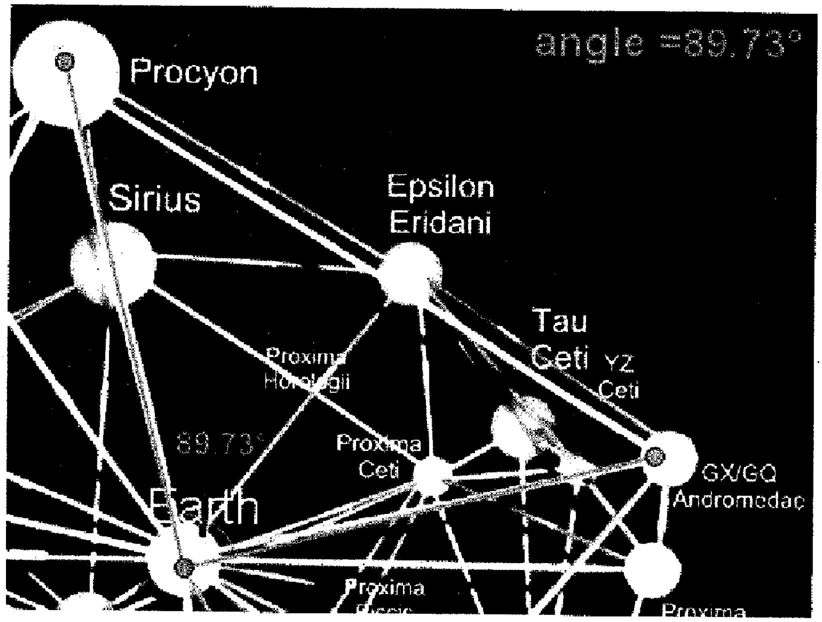
Procedures

During the first part of our lesson that focused on the implementation of the *My Theory* task, we went through the following instructional steps: 1) the introduction

Table 1: Distances of the star to the Earth

Name of stars	Distance from the Earth (light year)
Alpha Centauri	4.3
Barnard's	5.9
Sirius	8.7
Epsilon Eridani	10.7
Proxima Cygni	11.1
Epsilon Indi	11.3
Procyon	11.4

Figure 5. A snapshot from the applet



of the topic, 2) small group-work for the *My Theory* Problem 1, 3) whole class discussion about *My Theory* Problem 1, 4) small group-work for the *My Theory* Problem 2, and 5) whole class discussion about *My Theory* Problem 2 and the conclusion. During the introduction part, we first informed students about our goals and learning objectives for the lesson. Next, we refreshed students' memory about algebraic identities, trigonometric ratios, and the Pythagorean Theorem. For example, we covered the following mathematical principles during the introduction by orchestrating a whole class discussion (see Figure 6).

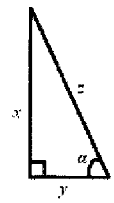
After the introduction, the students worked in their small groups to solve the *My Theory* Problem 1. During the whole class discussion, small groups who used algebraic identities or trigonometric ratios with some errors were initially asked to participate so that other students could contribute to the discussion for corrections. Due to the complexity of the *My Theory* Problem 2, we intentionally scaffolded students more during their small group work. In the conclusion part, we distributed the *My Theory* Solution Sheet and briefly described the Star Map task for the for next class hour.

We organized the second part of our lesson by 1) reviewing the law of cosines, 2) describing the Star Map task, and 3) wrapping up. Before students were assigned to work on the Student Worksheet Part 2 (i.e., the Star Map task), the first author

Figure 6. Major Mathematical Principles covered during the introduction of our lesson

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

$$x = z \cdot sin[\alpha]$$
$$y = z \cdot cos[\alpha]$$
$$(sin[\alpha])^2 + (cos[\alpha])^2 = 1$$



helped the students recall the law of cosines from the first part of the lesson. After the students completed their small group-work, we collected their Student Worksheets to inform the cooperative teacher about the students' learning gains from our intervention.

To wrap up our lesson, we presented the history of the law of cosines and its relationship with Earth and space sciences. More specifically, we described Euclid's way of proving the law of cosines and his contributions to spherical astronomy and Al Battani's conception of *angular distance* to measure the distances between stars in their times. Finally, to pique the students' interest in Earth and space sciences, we shared a YouTube Video (BrainStuff - HowStuffWorks, 2013) about the methods astronomers use to measure distances in space.

Extensions

We developed an extension of the Star Map task for those who finished the actual task earlier than the rest of the class. In the extension, we asked students to find the angle between two stars. To find the answer for the extension, students needed to use the inverse trigonometric function feature on their calculators. Students were then asked to check the same angle approximately with the applet:

If you know that the distance between Proxima Cygni and Barnard's stars is 9 light years, what should be the angular distance (angle) between these stars (look at the table of distance)?

CHALLENGES BASED ON STUDENT REFLECTIONS

At the end of our lesson, we collected all of the student worksheets and additionally asked students to fill out a two-question reflection form. The student reflections were one of the ultimate products of our lesson. The reflections provided student feedback on their views about instructional choices and their interest in learning about Earth and space sciences while covering the law of cosines. In other words, we administered student reflections to improve the quality of our lesson for revisions and future implementations. The following two questions were included in the student reflections form:

1. Did you see a connection between *My Theory* Problem 1 and Problem 2 in *My Theory* task? Do you think this pattern (connection between Problem 1 and 2) helped your construction of the law of cosines? How? If it did not, why?
2. Did the Star Map task and the use of Earth and space sciences as a theme pique your interest and motivate you in mathematics? Why or why not? Please explain.

Six out of seventeen students algebraically constructed the law of cosines. Whereas all students answered the *My Theory* Problem 1 question correctly, approximately two thirds of the class failed to algebraically construct the law of cosines. In order to facilitate the students' law construction, we applied a pattern from a concrete example to an abstract example in the *My Theory* task. It seems like the pattern of the task (i.e., from concrete to abstract), the explicit use of the mathematics language in the task, and the use of their previous knowledge helped some students construct the law of cosines. Thirteen out of seventeen students claimed that they saw the connection, and some students reasoned that the pattern helped them understand the law of cosines.

Even though most students claimed that they saw the connection between the concrete and abstract problems, as we mentioned before, eleven students could not construct the law of cosines by themselves. We speculate that ninety minutes was not a sufficient amount of time for students to develop a geometrical principle on their own. Many students discussed within their reflections that they needed to have more time to construct the law of cosines. Moreover, the student worksheets revealed that many students could not execute the algebraic manipulations appropriately. Students additionally indicated that they were not accustomed to producing proofs in their geometry class. Since our lesson was one of their early experiences with developing a proof for a geometrical principle, students needed to spend more time with the *My Theory* task.

Thirteen out of seventeen students liked the Star Map task and astronomy theme for the lesson due to its relationship to real life and the opportunity to represent mathematics untraditionally. Five students did not like the Star Map task due to their lack of interest in Earth and space sciences, unclarity of the relationship between the theory and practice, and overall difficulty of the task.

We may conclude from this experience that representing a pattern, through the use of a concrete example followed by an abstract example, can be helpful in developing students' understanding of mathematical proofs. However, the students' knowledge and experience with trigonometric ratios, algebraic manipulations, and mathematical proofs might be influential factors in determining the effectiveness of instruction. Furthermore, the majority of the students liked the context of astronomy for mathematics. The Star Map activity provided students with an opportunity to recognize the use of mathematics in real life. Regarding students' reflections, mathematics teachers should find similar ways to integrate science and technology content into their mathematics instruction. By observing the practical applications, students feel more engaged and motivated to learn advanced mathematical concepts.

RECOMMENDATIONS: HOW TO STEAM MORE

In this chapter, we presented a STEAM Education case for which mathematical content was the focus. As our lesson was at the Content Specific STEAM Education level, further efforts to make the disciplines more integrative are needed. To begin with, we consider the collaborative design and development of lessons, by teachers who are highly qualified in various subject areas, to promote the integration of the STEAM fields. Our lesson was only developed by mathematics teachers, including the authors of this chapter and the cooperating teacher; no science or computer science teacher was involved during the lesson development process. The gains from a collaboration among teachers having expertise in different STEAM fields would be twofold. First, secondary science and computer science teachers would rigorously contribute to the development of the students' scientific and computational thinking skills. Second, such a collaboration among multiple subject-matter teachers would enable students to perceive the lesson from a multidisciplinary standpoint instead of perceiving the lesson as an isolated mathematical principal.

Our lesson provided students with a ready-made applet so students would not need to spend their time applying computational thinking to develop an applet that would help them solve the given problem independently. An alternative lesson design in which students are guided to openly use dynamic geometry software during the problem solution process could also benefit students by allowing them to practice their computational thinking and technology skills.

The emphasis of engineering and art content in our lesson was limited. To amplify these two disciplines, we consider the use of data collection, analysis, and modelling techniques as the main instructional components. The "NASA's Open Data Portal" (n.d.) could enable teachers and students to get access to any data set of interest free of charge. After identifying the data and conducting some analyses on it, students could be asked to develop a 3D model to demonstrate their unique Star Map. That way, all letters within the STEAM acronym could be touched upon to some extent. We still anticipate possible revisions of our lesson to further integrate the disciplines of art and engineering, and to promote students' computational and problem-solving skills around a focused content.

Future efforts and detailed conceptions of STEAM Education should aim to elevate Content Specific implementations to a Universal STEAM curriculum, in which lessons are independent from any focused content/discipline. Developing a Universal STEAM curriculum should be the priority of teachers, teacher educators, and curriculum developers.

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KEY TERMS AND DEFINITIONS

Algebraic Identities: Set of equalities that are held valid for any group of variables. The standard algebraic product identity can be established using the Binomial Theorem. A subset of these equalities can also be used to find the factors for given formulas.

Angular Distance: The measurement of the angle between two objects based on the use of optical instruments by an observer. While angle on a 2D plane can be measured using protractor, identifying the angular distance between two objects in 3D space requires the use of angular distance by an observer.

Dynamic Geometry Software: Instructional technology commonly used for the geometry content. The affordances of this technology enables users to construct various geometrical figures, develop conjectures and assess the validity of these conjectures. GeoGebra is an example of the dynamic geometry software.

Information Communication Technologies: This is the broader term used for infrastructure and devices used for communication, application, collection and analysis of data.

Law of Cosines: This is the equation used to find one unknown side of any triangle when the lengths of the other two vertices and the angle between these two vertices are given.

Pythagorean Theorem: This theorem describes the numerical relationship among three vertices of any right triangle.

Trigonometric Ratios: Trigonometric relationships in any right triangle. There are 3 basic trigonometric ratios: sine, cosine, and tangent. This list can be expanded with the following additional trigonometric ratios: cotangent, secant, and cosecant.

ENDNOTE

[1] Solution sheet was distributed at the end of the class meeting.