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A Note on Hurwitzian Numbers

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Abstract. In this note Hurwitzian numbers are defined for the nearest integer, and backward continued fraction expansions, and Nakada's α-expansions. It is shown that the set of Hurwitzian numbers for these continued fractions coincides with the classical set of such numbers.

1. Introduction.

It is well-known that every real irrational number x has a unique regular continued fraction expansion of the form

$$x = [a_0; a_1, \cdots, a_n, \cdots], \tag{1}$$

where $a_0 \in \mathbf{Z}$ is such that $x - a_0 \in [0, 1)$, and $a_n \in \mathbf{N}$ for $n \geq 1$. The number x is called *Hurwitzian* if (1) can be written as

$$x = [a_0; a_1, \cdots, a_n, \overline{a_{n+1}(k), \cdots, a_{n+p}(k)}]_{k=0}^{\infty}, \tag{2}$$

where $a_{n+1}(k), \cdots, a_{n+p}(k)$ (the so-called *quasi period* of x) are polynomials with rational coefficients which take positive integral values for $k = 0, 1, 2, \cdots$, and at least one of them is not constant. By the bar we mean that $a_{n+i+kp} = a_{n+i}(k)$, where $1 \leq i \leq p$ and $k \geq 0$. A well-known example of such numbers is $e = [2; \overline{1, 2k+2, 1}]_{k=0}^{\infty}$; see [P] for more examples. Hurwitzian numbers are generalizations of numbers with an eventually periodic continued fraction expansion. An old and classical result states, that a number x is a quadratic irrational (that is, an irrational root of a polynomial of degree 2 with integer coefficients) if and only if x has a continued fraction expansion which is eventually periodic, i.e., if x is of the form

$$x = [a_0; a_1, \cdots, a_p, \overline{a_{p+1}, \cdots, a_{p+\ell}}], \quad p \geq 0, \ \ell \geq 1, \tag{3}$$

where the bar indicates the period, see [HW], [O] or [P] for various classical proofs of this result.

Apart from the regular continued fraction (RCF) expansion of x there are very many other—classical—continued fraction expansions of x, such as the nearest integer continued fraction (NICF) expansion, the 'backward' continued fraction expansion, and Nakada's α-expansions. In this note we will define what Hurwitzian numbers are for such continued

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fraction expansions and show that their set of Hurwitzian numbers coincides with the classical set of Hurwitzian numbers. As a by-product quadratic irrationals will have an eventually period expansion for each of these expansions.

2. Hurwitzian numbers for the NICF.

Every $x \in \mathbf{R} \setminus \mathbf{Q}$ can be expanded in a unique continued fraction expansion

$$x = b_0 + \cfrac{e_1}{b_1 + \cfrac{e_2}{b_2 + \ddots + \cfrac{e_n}{b_n + \ddots}}} =: [\, b_0;\ e_1/b_1,\ e_2/b_2,\ \cdots,\ e_n/b_n,\ \cdots]\,,$$

satisfying $b_0 \in \mathbf{Z}$, $x - b_0 \in [-1/2, 1/2)$, $e_n = \pm 1$, $b_n \in \mathbf{N}$ and $e_{n+1} + b_n \geq 2$ for $n \geq 1$. This continued fraction expansion is known as the *nearest integer continued fraction* (NICF) expansion of x.

In [K] it is shown that the NICF expansion can be obtained from the RCF by singularizing the first, the third, etc. 1's in every block of consecutive 1's preceded by either a partial quotient different from 1 or preceded by a_0. This singularization process is based upon the identity

$$A + \cfrac{e}{1 + \cfrac{1}{B + \xi}} = A + e + \frac{-\,e}{B + 1 + \xi}\,.$$

EXAMPLE 1. The NICF expansion of e is given by

$$[3; -1/4,\ \overline{-1/2,\ 1/(2k+5)}]_{k=0}^{\infty}\,.$$

In view of this example we have the following definition.

DEFINITION 1. Let $x \in \mathbf{R} \setminus \mathbf{Q}$. Then x has an NICF-Hurwitzian expansion if

$$x = [b_0; e_1/b_1, \cdots, e_n/b_n, \overline{e_{n+1}/b_{n+1}(k), \cdots, e_{n+p}/b_{n+p}(k)}]_{k=0}^{\infty}$$

where $b_0 \in \mathbf{Z}$, $x - b_0 \in [-1/2, 1/2)$, $e_n = \pm 1$, $b_n \in \mathbf{N}$ and $e_{n+1} + b_n \geq 2$ for $n \geq 1$. Moreover, for $i = 1, \cdots, p$ we have that $b_{n+i}(k)$ are polynomials with rational coefficients which take positive integral values for $k = 0, 1, 2, \cdots$, and at least one of them is non-constant.

The following result gives the necessary and sufficient condition for an irrational number to have an NICF-Hurwitzian expansion.

THEOREM 1. *Let $x \in \mathbf{R} \setminus \mathbf{Q}$. Then x is Hurwitzian if and only if x has an NICF-Hurwitzian expansion.*

PROOF. Let x be a Hurwitzian number with RCF expansion given by (1) and (2). Let $m_0 \in \mathbf{N}$, $m_0 \geq n$, be such that $a_{m_0} > 1$. Note that (2) can also be written as

$$x = [a_0; a_1, \cdots, a_{m_0}, \overline{\tilde{a}_{m_0+1}(k), \cdots, \tilde{a}_{m_0+p}(k)}]_{k=0}^{\infty}, \tag{4}$$

where $a_1, \cdots, a_{m_0}$ are positive integers, and where $\tilde{a}_{m_0+1}(k), \cdots, \tilde{a}_{m_0+p}(k)$ are polynomials with rational coefficients which take positive integral values for $k = 0, 1, 2, \cdots$, and at least one of them is not constant. Suppose moreover that m_0 is chosen in such a way, that for all $k \geq 0$ all the non-constant polynomials in the quasi-period $\tilde{a}_{m_0+1}(k), \cdots, \tilde{a}_{m_0+p}(k)$ have values greater than 1.

For $i \in \{1, \cdots, p-1\}$ we consider 2 cases:

Case (i): $a_{m_0+i} = 1$. By definition of a Hurwitzian number there exist numbers $j_1 \in \{0, 1, \cdots, i-1\}$ and $j_2 \in \{i+1, \cdots, p\}$ for which $a_{m_0+j_1} > 1$, $a_{m_0+j_2} > 1$, and

$$a_{m_0+j_1+1} = \cdots = a_{m_0+i} = \cdots = a_{m_0+j_2-1} = 1 .$$

In case $i - j_1$ is odd the digit $a_{m_0+i} = 1$ will be singularized, and in case $i - j_1$ is even it will not be singularized, but it will either change into $-1/2$ if $j_2 = i+1$, or into $-1/3$ if $j_2 \geq i+2$. Due to the quasi-periodicity and by definition of m_0 we have for each $k \in \mathbf{N}$ that

$$a_{m_0+j_1+kp+1} = \cdots = a_{m_0+i+kp} = \cdots = a_{m_0+j_2+kp-1} = 1 ,$$

and each of these blocks is singularized in the same way as the block $a_{m_0+j_1+1} = \cdots = a_{m_0+i} = \cdots = a_{m_0+j_2-1}$ was singularized, which means the same thing will happen to $a_{m_0+i+(k-1)p} = 1$ for all $k \in \mathbf{N}$.

Case (ii): $a_{m_0+i} > 1$ (a_{m_0+i} is either a constant or a polynomial). We have 4 possible cases:

(a) $a_{m_0+i-1} = 1 = a_{m_0+i+1}$. In this case, $a_{m_0+i-1} = 1$ belongs to a block of 1's and will be singularized if and only if this block has odd length. On the other hand, $a_{m_0+i+1} = 1$ will always be singularized, so that a_{m_0+i} will either become $-1/(a_{m_0+i}+2)$ (if the block of 1's 'before' a_{m_0+i} has odd length), or become $1/(a_{m_0+i}+1)$.

(b) $a_{m_0+i-1} \neq 1 = a_{m_0+i+1}$. In this case, a_{m_0+i} becomes $1/(a_{m_0+i}+1)$, due to the singularization of $a_{m_0+i+1} = 1$.

(c) $a_{m_0+i-1} = 1 \neq a_{m_0+i+1}$. In this case, a_{m_0+i} becomes either $-1/(a_{m_0+i}+1)$, or remains unchanged, depending on whether $a_{m_0+i-1} = 1$ is singularized or not.

(d) $a_{m_0+i-1} \neq 1 \neq a_{m_0+i+1}$. In this case a_{m_0+i} will remain unchanged.

Due to the periodicity the same thing will happen to $a_{m_0+i+(k-1)p} > 1$ for all $k \in \mathbf{N}$.

To conclude, from (i) and (ii) we see that for each $i \in \{1, \cdots, p\}$ and for all $k \in \mathbf{N}$ one has exactly one of the following possibilities:

— $a_{m_0+i+(k-1)p} = 1$ always disappears due to a singularization;

— $a_{m_0+i+(k-1)p} > 1$ always remains unchanged;

— $a_{m_0+i+(k-1)p} > 1$ always becomes $-1/(a_{m_0+i+(k-1)p}+1)$ due to the singularization of a digit 1 before it;

— $a_{m_0+i+(k-1)p} = 1$ always becomes $1/(a_{m_0+i+(k-1)p} + 1)$ due to the singularization of a digit 1 after it;

— $a_{m_0+i+(k-1)p} = 1$ always becomes $-1/(a_{m_0+i+(k-1)p} + 2)$ due to the singularization of a digit 1 before and after it.

Thus we obtain a quasi-period for the NICF expansion of x.

Conversely, since the singularization process can be reversed in a unique way, we see that a NICF-Hurwitzian number x is also Hurwitzian. □

Applying the procedure given in the proof of Theorem 1 yields that the NICF-expansion of e is given by $e = [3; -1/4,\ -1/2,\ \overline{1/(2k+5),\ -1/2}]_{k=0}^{\infty}$, which is another way of writing e in Example 1.

From the proof of Theorem 1 it is at once clear that x is a quadratic irrational if and only if the NICF-expansion of x is eventually periodic.

3. Hurwitzian numbers for the backward continued fraction.

Every $x \in \mathbf{R} \setminus \mathbf{Q}$ can be expanded in a unique continued fraction expansion

$$c_0 - \cfrac{1}{c_1 - \cfrac{1}{c_2 - \ddots - \cfrac{1}{c_n - \ddots}}} =: [\, c_0;\ -1/c_1,\ -1/c_2,\ \cdots,\ -1/c_n,\ \cdots],$$

where $c_0 \in \mathbf{Z}$ such that $x - c_0 \in [-1, 0)$ and c_i's are all integers greater than 1. This continued fraction is known as the *backward continued fraction* expansion of x; see [DK] for details.

Proposition 2 in [DK] gives an algorithm yielding the backward continued fraction expansion from the regular one using singularizations and insertions. The latter is based on the following identity.

$$A + \frac{1}{B+\xi} = A + 1 + \cfrac{-1}{1 + \cfrac{1}{B - 1 + \xi}}.$$

From this algorithm it follows that $x = [a_0; a_1,\ a_2,\ \cdots]$ has as backward expansion

$$[a_0 + 1;\ (-1/2)^{a_1 - 1},\ -1/(a_2 + 2),\ (-1/2)^{a_3 - 1},\ -1/(a_4 + 2),\ \cdots] \tag{5}$$

where $(-1/2)^t$ is an abbreviation of $\underbrace{-1/2, \cdots, -1/2}_{t\text{-times}}$ for $t \geq 1$. In case $t = 0$, the term $(1/2)^t$ should be omitted.

EXAMPLE 2. The backward expansion of e is given by

$$[3;\ \overline{-1/(4k+4),\ -1/3,\ (-1/2)^{4k+3},\ -1/3}\,]_{k=0}^{\infty}.$$

This example leads to the following definition.

DEFINITION 2. Let $x \in \mathbf{R} \setminus \mathbf{Q}$. Then x has a backward-Hurwitzian expansion if

$$x = [c_0;\ (-1/c_1)^{r_1}, \cdots, (-1/c_n)^{r_n}, \overline{(-1/c_{n+1}(k))^{r_{n+1}(k)}, \cdots, (-1/c_{n+p}(k))^{r_{n+p}(k)}}\ ]_{k=0}^{\infty}$$

where $c_0 \in \mathbf{Z}$ such that $x - c_0 \in [-1, 0)$; $(c_i, r_i) = (c, 1)$ or $(2, r)$ for $i = 1, \cdots, n$, where c is an integer greater than 2 and r a positive integer. We call p the 'length' of the quasi-period. Moreover,

$$(c_{n+i}(k), r_{n+i}(k)) = (f_i(k), 1) \quad \text{or} \quad (2, g_i(k))$$

for $i = 1, \cdots, p$ where $f_i(k)$ and $g_i(k)$ are polynomials with rational coefficients which take positive integral values for $k = 0, 1, 2, \cdots$ and at least one of them is not constant. Here $(-1/c)^r$ is an abbreviation of $\underbrace{-1/c, \cdots, -1/c}_{r\text{-times}}$.

The following result gives the necessary and sufficient condition for an irrational number to have a backward-Hurwitzian expansion.

THEOREM 2. *Let $x \in \mathbf{R} \setminus \mathbf{Q}$. Then x is Hurwitzian if and only if x has a backward-Hurwitzian expansion.*

PROOF. Let x be a Hurwitzian number, with RCF-expansion (1). We first note that (5) yields that a_n in the RCF-expansion of x becomes $(-1/2)^{a_n-1}$ in the backward expansion of x if n is odd, and becomes $-1/(a_n + 2)$ if n is even. Let m_0 be defined as in the proof of Theorem 1. Then for all $i > m_0$ we observe the following:

(i) If $a_i = 1$, then it either disappears in case i is odd, or becomes $-1/3$ in case i is even.

(ii) If $a_i > 1$, then it either becomes $(-1/2)^{a_i-1}$ in case i is odd, or $-1/(a_i + 2)$ in case i is even.

Let p be the length of the quasi-period of the RCF-expansion of x. We see that for all $k \in \mathbf{N}$ the same thing will happen to each $a_{i+(k-1)p}$ if p is even or to each $a_{i+2(k-1)p}$ if p is odd, which yields a quasi-periodicity for the backward expansion of x.

Conversely, since the singularization and insertion processes can be reversed in a unique way, we see that a backward-Hurwitzian number x is also Hurwitzian. □

Clearly x is a quadratic irrational if and only if the backward-expansion of x is eventually periodic. The next section gives a generalization of Section 2.

4. Hurwitzian numbers for α-expansions.

In this section we will define Hurwitzian numbers for the so-called α-expansions, of which the nearest interger continued fraction expansion is an example. These α-expansions were introduced and studied by H. Nakada in 1981 ([N]). We will show that Hurwitzian numbers for these α-expansions also coincide with the classical Hurwitzian numbers.

For $\alpha \in [1/2, 1]$, let $x \in [\alpha - 1, \alpha]$ and define

$$
\begin{aligned}
f_1 &= f_1(x) := \lfloor |1/x| + 1 - \alpha \rfloor , \quad x \neq 0 , \\
f_n &= f_n(x) := f_1(T_\alpha^{n-1}(x)) , \quad n \geq 2 , \quad T_\alpha^{n-1}(x) \neq 0 ,
\end{aligned} \tag{6}
$$

where $T_\alpha : [\alpha - 1, \alpha] \to [\alpha - 1, \alpha]$ is defined by

$$T_\alpha = |1/x| - \lfloor |1/x| + 1 - \alpha \rfloor$$

and $\lfloor \xi \rfloor$ denotes the largest integer not exceeding ξ.

Every $x \in \mathbf{R} \setminus \mathbf{Q}$ can be expanded in a continued fraction expansion

$$x = [f_0; e_1/f_1, e_2/f_2, \cdots , e_n/f_n, \cdots] ,$$

where $f_0 \in \mathbf{Z}$, $x - f_0 \in [\alpha - 1, \alpha)$, $e_n = \pm 1$, $f_n \in \mathbf{N}$, $n \geq 1$, are given by (6). We call this continued fraction the α-*expansion* of x.

REMARK. Note that for $\alpha = 1/2$ one has the NICF-expansion, while $\alpha = 1$ is the RCF case.

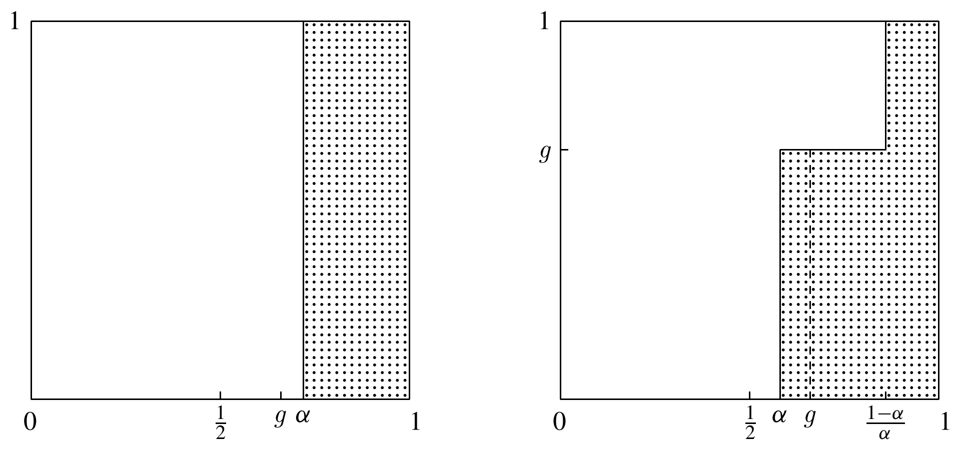
In [K] it is shown that α-expansions can be viewed as S-expansions, with singularization areas

$$S_\alpha = [\alpha, 1] \times [0, 1] , \quad \text{if} \quad g < \alpha \leq 1$$

and

$$S_\alpha = [\alpha, g) \times [0, g) \cup [g, (1-\alpha)/\alpha] \times [0, g] \cup ((1-\alpha)/\alpha, 1] \times [0, 1]$$

in case $1/2 \leq \alpha \leq g$, where $g = (\sqrt{5} - 1)/2$; see Figure 1.



(i) $g < \alpha \leq 1$ (ii) $1/2 \leq \alpha \leq g$

FIGURE 1. Singularization areas for α-expansions.

In general a singularization area S is a subset of the so-called natural extension $[0,1) \times [0,1]$ of the RCF-expansion, which satisfies the following three conditions:

(i): $S \subset [1/2,1) \times [0,1]$; (ii): $\mathcal{T}(S) \cap S = \emptyset$ and (iii): $\lambda(\partial S) = 0$.

Here λ is Lebesgue measure on $[0,1) \times [0,1]$, and $\mathcal{T} : [0,1) \times [0,1] \to [0,1) \times [0,1]$ is the natural extension map of the RCF-expansion, given by

$$\mathcal{T}(x,y) = \left(\frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, \frac{1}{\lfloor \frac{1}{x} \rfloor + y}\right),\ (x,y) \in (0,1) \times [0,1];\ \mathcal{T}(0,y) = (0,0),\ y \in [0,1]\,.$$

Let $x \in [0,1)$, with RCF-expansion $[a_0; a_1, a_2, \cdots]$. Then the S-expansion of x is obtained via the following algorithm:

$$\text{singularize } a_{n+1} = 1 \text{ if and only if } (T_n, V_n) \in S_\alpha\,,$$

where $T_n = [0; a_{n+1}, a_{n+2}, \cdots]$ and $V_n = [0; a_n, \cdots, a_1]$, i.e., $(T_n, V_n) = \mathcal{T}^n(x,0)$, for more details, see [K].

The following lemma is very handy.

LEMMA 1. *Let $x, y \in [0,1)$, with RCF-expansions*

$$x = [0; a_1(x), a_2(x), \cdots]\,, \quad y = [0; a_1(y), a_2(y), \cdots]\,.$$

Let $x \neq y$ and $k \in \mathbf{N} \cup \{0\}$ be such that

$$a_1(x) = a_1(y),\ \cdots,\ a_{k-1}(x) = a_{k-1}(y)\,, \quad \textit{and} \quad a_k(x) \neq a_k(y)\,.$$

Then one has

$$x > y \quad \textit{if and only if} \quad \begin{cases} a_k(x) < a_k(y) & \textit{if } k \textit{ is odd}\,, \\ a_k(x) > a_k(y) & \textit{if } k \textit{ is even}\,. \end{cases}$$

PROOF. For $n \in \mathbf{N}$, $a_1, \cdots, a_n \in \mathbf{N}$, define cylinders $\Delta_n(a_1, \cdots, a_n)$ by

$$\Delta_n(a_1, \cdots, a_n) = \{x \in [0,1)\,;\ a_1(x) = a_1, \cdots, a_n(x) = a_n\}\,.$$

For $x,\ y \in \Delta_{k-1}(a_1, \cdots, a_{k-1}),\ x < y$, one has by definition of the RCF-map $T = T_1$ that $T(x),\ T(y) \in \Delta_{k-2}(a_2, \cdots, a_{k-1})$, and $T(x) > T(y)$. Repeating this argument $k-2$-times, we find that $T^{k-2}(x),\ T^{k-2}(y) \in \Delta_1(a_{k-1})$, and that $T^{k-2}(x) < T^{k-2}(y)$ if and only if k is even. Since $T(\Delta_1(a_{k-1})) = [0,1)$ and $a_k(x) \neq a_k(y)$, it follows from the definition of T that $T^{k-1}(x) > T^{k-1}(y)$ if and only if k is even. Since $T^{k-1}(x) \in \Delta_1(a_k(x)) = (1/(a_k+1), 1/a_k]$, and $T^{k-1}(y) \in \Delta_1(a_k(y))$, it follows that $a_k(x) < a_k(y)$ if and only if k is even. □

We now define Hurwitzian numbers for α-expansions.

DEFINITION 3. Let $x \in \mathbf{R} \setminus \mathbf{Q}$. Then, for a fixed $\alpha \in [1/2,1]$, x has an α-Hurwitzian expansion if

$$x = [f_0; e_1/f_1, \cdots, e_n/f_n, \overline{e_{n+1}/f_{n+1}(k), \cdots, e_{n+p}/f_{n+p}(k)}]_{k=0}^{\infty} \tag{7}$$

is the α-expansion of x, where $f_0 \in \mathbf{Z}$, $x - f_0 \in [\alpha-1, \alpha)$, $e_n = \pm 1$, $f_n \in \mathbf{N}$, $n \geq 1$, are given by (6). Moreover, for $i = 1, \cdots, p$ we have that f_{n+i} are polynomials with rational coefficients which take positive integral values for $k = 0, 1, 2, \cdots$, and at least one of them is non-constant.

We have the following theorem.

THEOREM 3. *Let $x \in \mathbf{R}\backslash\mathbf{Q}$. Then x is Hurwitzian if and only if x has an α-Hurwitzian expansion.*

PROOF. As in the proof of Theorem 1, let $m_0 \in \mathbf{N}$ be such that $a_{m_0} > 1$, and for all $m \geq m_0$ all the non-constant polynomials in the quasi-period $\tilde{a}_{n+1}(k), \cdots, \tilde{a}_{n+p}(k)$ of the RCF-expansion (4) of x have values greater than 1. Let $k \in \{m_0 + 1, \cdots, m_0 + p\}$ be such that $a_k = 1$. Then

$$T^{k-1}(x) = [0; 1, a_{k+1}, \cdots].$$

CASE 1: $g < \alpha \leq 1$. In this case $a_k = 1$ must be singularized if and only if $T^{k-1}(x) \geq \alpha$.

Clearly there exists a minimal $i \in \{1, \cdots, p\}$ such that a_{k+i} is a value of a non-constant polynomial. Further, let $j \in \mathbf{N} \cup \{\infty\}$ be the first index such that

$$a_{k+j} \neq a_{j+1}(\alpha)$$

where $\alpha = [0; 1, a_2(\alpha), \cdots]$ is the RCF expansion of α.

In case $j \geq i$, there exists an $\ell_0 \geq 0$ such that, by Lemma 1 for all $\ell \geq \ell_0$

$$T^{k+\ell p-1}(x) > \alpha \quad \Leftrightarrow \quad i \text{ is odd},$$

implying that $a_{k+\ell p} = 1$ must be singularized for all $\ell \geq \ell_0$ if and only if i is odd. Otherwise, they are never singularized for all $\ell \geq \ell_0$.

If $1 \leq j \lneqq i$, then a_{k+j} is a constant different from $a_{j+1}(\alpha)$, so

$$T^{k+\ell p-1}(x) \geq \alpha \quad \Leftrightarrow \quad j \text{ is odd and } a_{k+j} > a_{j+1}(\alpha),$$

and we see that $a_{k+\ell p} = 1$ must be singularized for all $\ell \geq 0$ if and only if j is odd and $a_{k+j} > a_{j+1}(\alpha)$ (or equivalently, if and only if j is even and $a_{k+j} < a_{j+1}(\alpha)$).

CASE 2: $1/2 \leq \alpha \leq g$. In this case we have to consider (T_{k-1}, V_{k-1}). It is clear that there exists a minimal $h \in \{1, \cdots, p\}$ such that $a_{k+\ell p-h} > 1$ for all $\ell \geq 0$. If h is odd implying $V_{k-1} < g$, then $a_k = 1$ must be singularized if and only if $T_{k-1} > \alpha$. In this case, let i and j be defined as in Case 1. If $j \geq i$, then there exists an $\ell_2 \geq 0$ such that for all $\ell \geq \ell_2$ one has

$$T_{k+\ell p-1}(x) > \alpha \quad \Leftrightarrow \quad i \text{ is odd}.$$

If $1 \leq j \lneqq i$, one has

$$T_{k+\ell p-1} \geq \alpha \quad \Leftrightarrow \quad j \text{ is odd and } a_{k+j} > a_{j+1}(\alpha).$$

On the other hand, if h is even implying $V_{k-1} > g$, then $a_k = 1$ must be singularized if and only if $T_{k-1} > (1-\alpha)/\alpha$. Again let i be defined as in Case 1, but j be such that

$$a_{k+j} \neq a_{j+1}((1-\alpha)/\alpha),$$

where $[0;\, 1, a_2(\frac{1-\alpha}{\alpha}), a_2(\frac{1-\alpha}{\alpha}), \cdots]$ denotes the RCF expansion of $(1-\alpha)/\alpha$.
If $j \geq i$, then there exists an $\ell_3 \geq 0$ such that for all $\ell \geq \ell_3$ one has

$$T_{k+\ell p-1} > (1-\alpha)/\alpha \quad \Leftrightarrow \quad i \ \text{ is odd}\,.$$

If $1 \leq j \lneqq i$, one has

$$T_{k+\ell p-1} \geq (1-\alpha)/\alpha \quad \Leftrightarrow \quad j \ \text{ is odd and } a_{k+j} > a_{j+1}((1-\alpha)/\alpha). \qquad \square$$

EXAMPLE 3. Here we give α-expansions of e for some values of α.
(i) For $\alpha = 0.7$,

$$e = [3;\, -1/3,\, 1/2,\, \overline{-1/(2k+5),\, 1/2}]_{k=0}^{\infty}\,.$$

(ii) For $\alpha = 0.52$,

$$\begin{aligned} e = [3;\, &-1/4,\ -1/2,\ 1/5, \\ &-1/2,\ 1/7,\ -1/2,\ 1/9,\ -1/2,\ 1/10,\ 1/2,\ \overline{-1/(2k+13),\ 1/2}]_{k=0}^{\infty}\,. \end{aligned}$$

(iii) For $\alpha = 0.53$,

$$e = [3;\, -1/4,\ -1/2,\ 1/5,\ -1/2,\ 1/6,\ 1/2,\ \overline{-1/(2k+9),\ 1/2}]_{k=0}^{\infty}\,.$$

REMARKS. 1. From the proof of Theorem 3 it is at once clear that x is a quadratic irrational if and only if the α-expansion of x is eventually periodic.

2. Analogous to Definitions 1 and 3 we can define S-Hurwitzian number for any S-expansion. In case the singularization-area is 'nice' (such as the singularization-areas for Nakada's α-expansion, or for Minkowski's diagonal continued fraction expansion, see [H]), one can show that being S-Hurwitzian is equivalent to being Hurwitzian. However, it is possible to find singularization-areas S and numbers x such that x is Hurwitzian, but not S-Hurwitzian. Consider for example the following singularization-area S:

$$S = \bigcup_{p \text{ prime}} (2p+2, 2p+1] \times \left(\frac{1}{2}, 1\right).$$

One easily convinces oneself that e does not have an S-expansion which is S-Hurwitzian.

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