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AN EMPTY NUMBER LINE TO DEVELOP MENTAL ARITHMETIC STRATEGIES

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Abstract

Introducing the algorithm strategy prematurely without sufficient understanding of numbers could lead to student's innumeracy, while constructing mental arithmetic strategies in a realistic approach is suggested as an alternative. Furthermore, an empty number line is found to be a powerful model to do mental arithmetic strategies flexibly. Therefore, the present research attempts to answer the question about the development of student's learning process in constructing mental arithmetic strategies on an empty number line to solve addition problems. The question was answered by taking into account a design research methodology with two underpinning theories, i.e. (1) the Realistic Mathematics Education theory as a guide to develop a local instruction theory in a specific mathematical domain; and (2) the socio-constructivist analysis of instruction in which student's learning processes were viewed from both the individual and the social perspective. The result shows that the use of context, model, students' active contributions, and the role of the teacher could enhance students thinking in developing mental arithmetic strategies on an empty number line. However, an empty number line could also become meaningless for the development of student's mental arithmetic strategies when the significance of number relations is forgotten in the classroom discussion.

Keywords*: mental strategies, number line*

INTRODUCTION

In recent years, researchers in mathematics education have become increasingly interested in mental arithmetic as a new breakthrough that must precede algorithm in doing calculation for elementary school children (Treffers, 1991; Beishuizen, 1993; Reys *et al*, 1995; Klein, 1998). Mental calculation is required while dealing with daily life situation: for instance, it is meaningless to perform a pen and paper algorithm to calculate the amount of money needed to pay for two kilos of sugar and a box of chocolate in the market. Treffers (1991) argued that algorithm is one of the causes of innumeracy in primary schools when it is taught prematurely without context problems, while mental arithmetic strategies and estimation in a realistic approach are suggested as an alternative.

However, mental arithmetic strategies must be introduced thoughtfully with rich contextual situations, in which children have freedom to develop their understanding under the guidance of the teacher. In addition to this, Gravemeijer (1994a) pointed out that an empty number line is found to be a powerful model to do mental arithmetic strategies flexibly and to foster the development of more sophisticated strategies, but which could represent children's informal strategies at the same time.

Moreover, Klein (1998) came to the conclusion that providing children with a powerful model like the empty number line, establishing an open classroom culture in which children's solutions are taken seriously, and making teachers aware of both cognitive and motivational aspects of learning, will help every student become a flexible problem solver. Therefore, contextual situations, the use of model, the proactive role of the teacher and the classroom culture play a crucial role in the development of students' learning in a classroom community.

Although considerable research has been done in many countries, problems still remains in Indonesia, where the mathematics elementary school curriculum's objectives focus on the use of algorithm to solve double-digit addition and subtraction problems since the first grade. Furthermore, Indonesian classroom culture tends to perceive a classroom community with teachers as the central point and children are expected to be obedient in every manner. This will lead to the limitation of children's freedom in doing mathematics in their own way.

Aiming at the development of theory and improvement of practice about both the process of learning and means designed to support that learning, hence the present research attempts to answer the question about the development of student's learning process in constructing mental arithmetic strategies on an empty number line to solve addition problems. The question will be answered by taking into account a design research methodology with two underpinning theories, i.e. (1) the Realistic Mathematics Education theory (Freudenthal, 1991; Treffers, 1987; Gravemeijer, 1994b) as a guide to develop a local instruction theory in a specific mathematical domain; and (2) the socio-constructivist analysis of instruction (Cobb & Yackel, 1996; Cobb *et al*, 2001) where children's learning processes are viewed from both the individual perspective and the social perspective.

RME IN INDONESIA

The traditional approach in teaching and learning of mathematics in Indonesian schools put teachers in a central role where students become passive learners without a great deal of thought in doing mathematics. As a result, students often have difficulties to comprehend mathematical concepts and to construct and solve mathematical representation from a contextual problem, and the teaching style makes mathematics more difficult to learn and to understand.

On the other hand, a progressive innovation program, i.e. PMRI (Pendidikan Matematika Realistik Indonesia), that has been running for several years, has a primary aim to reform mathematics education in Indonesia. This innovation program is adapted from RME (Realistic Mathematics Education) in the Netherlands that views *mathematics as a human activity* (Freudenthal, 1991) in which students build their own understanding in doing mathematics under the guidance of the teacher. In contrast to traditional mathematics education that used a ready-made mathematics as a starting point for instruction, RME emphasizes mathematics education as a process of doing mathematics in reality that leads to a result, mathematics as a product. Sembiring, *et al* (2008) summarized from all RME studies in Indonesia that the RME approach could be utilized in Indonesia and stimulate reform in mathematics education.

Five Characteristics of RME

Treffers (1987) has defined five characteristics for Realistic Mathematics Education:

Phenomenological exploration.
Freudenthal's view of mathematics as a human activity requires an extensive phenomenological exploration that is aimed at acquiring a rich collection of contextual situation in which mathematical activities take place. The contextual situation is not merely a word problem, but it has to deal with a reality that makes sense for children with different levels. It should provide an ample space for children to build their own understanding and serve as a basis for model development.

Using models and symbols for progressive mathematization.
The development from intuitive, informal, context bound notions towards more formal mathematical concepts is a gradual process of progressive mathematization. A variety of models, schemes, diagrams, and symbols encourages children in this gradual process. In the present research, an empty number line is chosen as the model that will emerge first as a *model of* a situation (measuring situation) then it develops as a *model for* solving addition and subtraction problems.
Using students' own construction and productions

Students' own constructions deal with their *actions,* while their own productions deal with their *reflection.* Students have freedom to construct their own path in learning under the teacher's guidance. In this case, the teacher should help students to build on their understanding from what students know.

Interactivity.
The learning process of students is not merely an individual process, but it is also a social activity where students build on their understanding through discussions, collective work reviews, evaluation of various constructions on various levels (comparing strategies), and explanation by the teacher. The interactivity means that students are also confronted with the constructions and productions of their fellows, which can stimulate them to shorten their learning path and to become aware of, in this case, more sophisticated strategies in solving addition and subtraction problems up to 100.

Intertwinement.
The mathematical domain in which students are engaged in should be considered related to other domains, the intertwining of learning strands. In this design research the 'calculation for numbers up to 100' through the measuring activity is intertwined with 'measuring conceptions'. Students develop their understanding of measuring conceptions as well as their counting strategies in the measuring context.

MENTAL ARITHMETIC STRATEGIES ON AN EMPTY NUMBER LINE

The mental arithmetic strategies defined in the present research aim at understanding number relations to perform mental calculation, not only bare number problems, but also contextual problem situations, and draws on Buys (in Van den Heuvel-Panhuizen, M. 2001) definition:

Mental arithmetic is a way of approaching numbers and numerical information in which numbers are dealt with in a handy and flexible way, and characterized by:

- *Working with numbers and not with digits*
- *Using elementary calculation properties and number relations*
- *Being supported by a well-developed feeling for numbers.*

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- *Possibly using suitable intermediate notes according to the situation, but mainly by calculating mentally.*

Furthermore, Buys described three elementary forms of mental arithmetic:

Mental arithmetic by a *stringing* strategy, in which keeping the first number as a whole and splitting the second number into tens and ones to be added to or subtracted from the first number. For example: 48 + 29 = (48+20)+9 (jump-of-ten) or 48 + 29 = (48+2)+20+7 (jump-via-ten)

Mental arithmetic by a *splitting* strategy, in which the numbers are broken down in tens and ones and processed separately. For example, see figure 1.

Mental arithmetic by a *varying* strategy is actually a combination of stringing and splitting by employing number relations, such as swapping (7+49 = 49+7), doubling (26+27 = [25+1]+[25+2]), using inverse relationships (the answer of 76−49 is found by adding on from 49), and compensating (74−38 = 74−40+2)

There are three levels of students' thinking in solving addition and subtraction problems up to 100 (figure 1). In the lowest level, students solve addition and subtraction problems by counting one by one. They still don't recognize structure of ten that could help them counting easier. In the next level, students acquire the idea of 'unitizing' which means they understand that a number of objects can build another unit. For example, one pack of T-shirt is twelve T-shirts; one pack of books is ten books. Therefore, students understand that 3 packs of books correspond to 30. At the top level, student should already master the previous concepts and strategies before they are taught algorithm. Most problems that happen in primary school is that teachers skip the 'floating capacity' of the ice berg and they go straight to algorithm. The teaching of algorithm is best after the 'calculation by structuring' is developed in student's thinking.

An empty number line -a number line that is presented with no numbers or markers on it- is chosen as a model to represent students' strategies during mental computation. By representing students' strategies on the empty number line, each step in students' thinking can be recorded. Therefore, it allows them to track errors. Moreover, the empty number line provides a visual representation of students' thinking that could engage the classroom community to share and discuss the most sophisticated strategies in solving addition and subtraction problems.

Figure 1 tells us the development of students' thinking as well as the emergence of an empty number line. The empty number line could be introduced through a string of beads that alternate in color every 10 beads in a measuring activity. The empty number line is emerged first as a *model of* a situation (measuring situation) then it develops as a *model for* solving addition and subtraction problems using mental arithmetic strategies. At the end, students are expected to be able to solve problems mentally by applying number relations without having to draw the number line.

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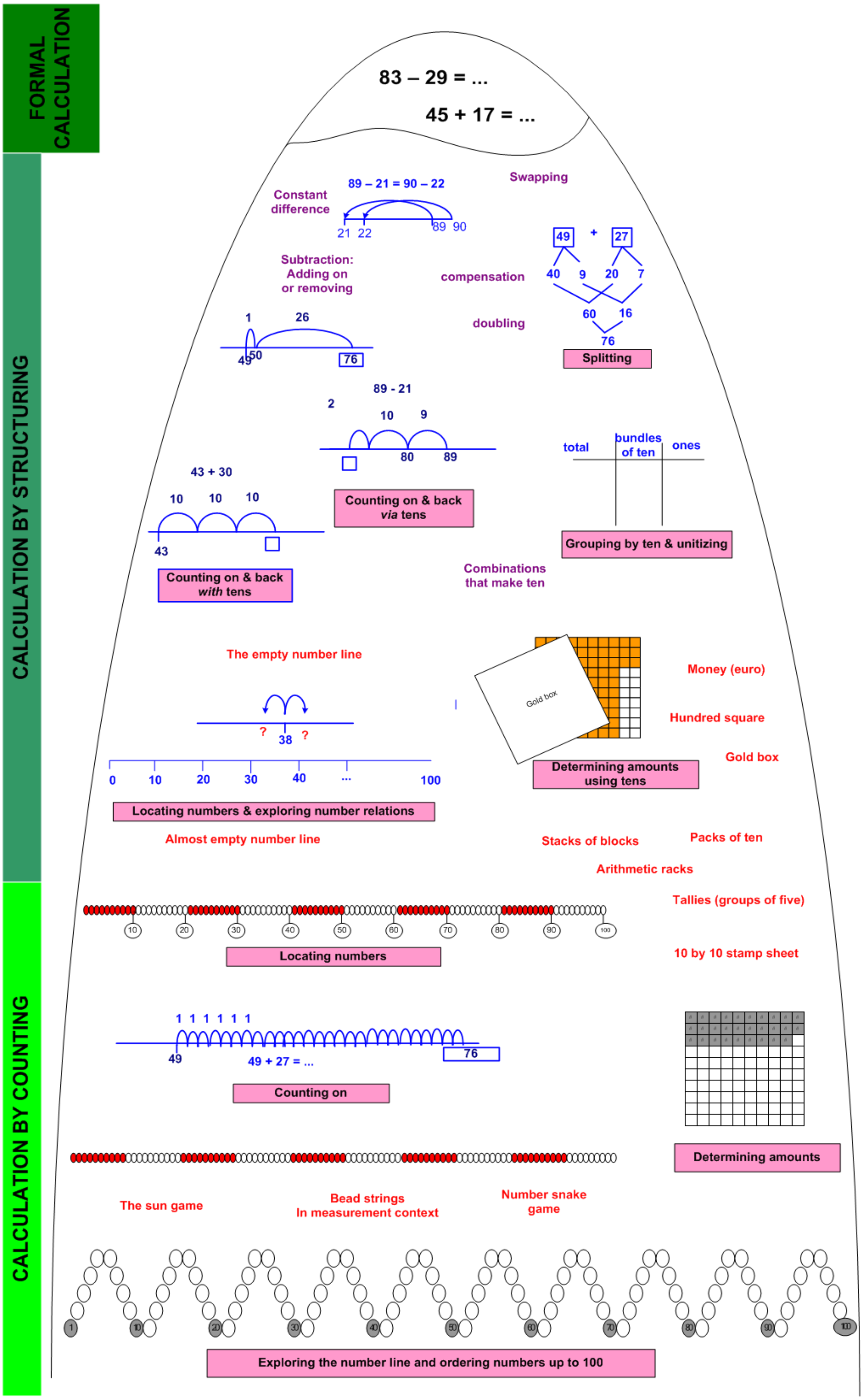

Figure 1. The Level of Student's Thinking in Solving Addition and Subtraction Problems

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METHODOLOGY

Our methodology falls under the general heading of "design research" which was first proposed as "developmental research" by Freudenthal in the Netherlands to develop the so-called domain-specific instruction theory of RME (Gravemeijer & Cobb, 2006; Freudenthal, 1991).

Developmental research means:
Experiencing the cyclic process of development and research so consciously, and reporting on it so candidly that it justifies itself, and this experience can be transmitted to others to become like their own experience. (Freudenthal, 1991: 161)

Basically, a design research has three essential phases, which are the design and preparation phase (thought experiment), the teaching experiment phase (instruction experiment), and the retrospective analysis phase (Gravemeijer & Cobb, 2006; Cobb *et al.*, 2003). Each of these forms a cyclic process both on its own and in a whole design research. Therefore the design experiment consists of cyclic processes of thought experiments and instruction experiments (Freudenthal, 1991).

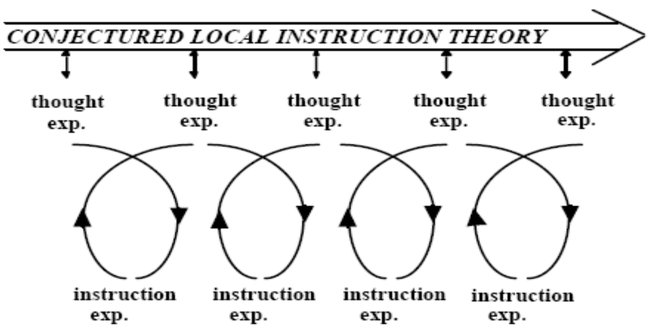

Figure 2. Reflexive relation between theory and experiments (Gravemeijer & Cobb, 2006)

In the first phase of this design research, a hypothetical learning trajectory is developed under the guidance of the domain-specific instruction theory RME, and then put to the test in the teaching experiment phase, and finally the conjectures are either proved or disproved in the analysis phase to reconstruct the local instruction theory. In this respect, the conjectured local instruction theory guides the cyclically teaching experiment phase while the experiment contributes to the development of the local instruction theory.

DATA ANALYSIS

In this design research, data such as video recordings of the classroom situation, students' works, and field notes were collected during the teaching experiment phase. The actual learning process is compared to the hypothetical learning trajectory that was developed in the first phase of the design research. In analyzing the data, Cobb *et al* (2001) describe criteria for analyzing the collective learning of a classroom community in terms of the evolution of classroom mathematical practices

It should enable us to document the collective mathematical development of the classroom community over the extended periods of time covered by instructional sequences

It should enable us to document the developing mathematical reasoning of individual students as they participate in the practices of the classroom community

It should result in analyses that provide feedback to inform the improvement of our instructional designs.

In this section, we only focus on the development of students of second graders in SDN Percontohan Kompleks IKIP, Jakarta, Indonesia with 37 children in constructing mental arithmetic strategies on an empty number line. Several data of students' works were chosen and analyzed.

Figure 3 shows us Yona's strategies in solving problem: "*In the competition for Indonesian Independence Day on the 17th of August, Annisa has to collect 50 hidden Indonesian flags. Annisa already has 19 flags with her. How many more flags does she have to find?*"

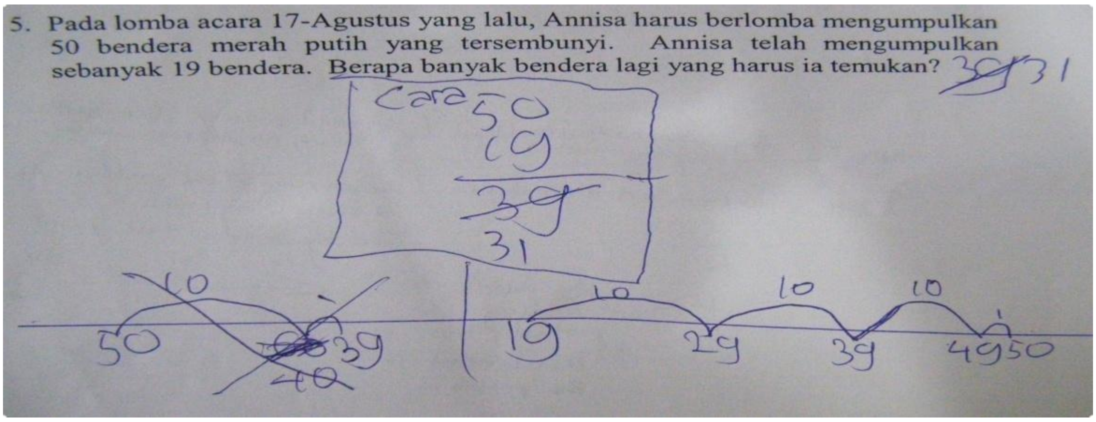

Figure 3 An empty number line is used to correct mistake when dealing with algorithm

As it is shown in figure 3, Yona makes a common mistake in performing the algorithm procedure. She subtracts 9 from 0 (the second digits) as she subtracts 0 from 9, then she subtracts 1 from 5 as if she borrows a ten for the second digit. Then, she draws an empty number line and writes 50 on the left side and draws a jump of ten to the right. At first, she though that she is adding 10 to 50, but she realizes that she is doing subtraction. Therefore, she crosses the 60 and 40 replaces the 60. She draws another smaller jump to go to the 39 (because she knows from the algorithm that the answer is 39). The mark on top of the smaller jump indicates that she was going to write number 1, but she suddenly stops and realizes that it is not supposed to be 1. When she adds the 10 and 1 together, she won't get 19 as the number that is subtracted from 50 in the problem. After that, she realizes her mistake and crosses her first number line. Then she gets 31 as the correct answer either by performing the algorithm once again (correcting 39 to be 31) or by calculating on the empty number line on the right side. If we observe thoroughly how Yona works on an empty number line (see figures 4 and 5), she is able to make a jump of 20 or 40 at once on an empty number line, while in fact she makes very careful jumps of tens in this case. She could have performed a jump of thirty at once if she already knew the answer from the algorithm (if she corrects the algorithm first). Therefore, she performs the calculation on the empty number line to correct her mistake in algorithm.

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Figure 4 shows us another strategy of Yona in solving problem: "*How do you solve this problem on an empty number line?*"

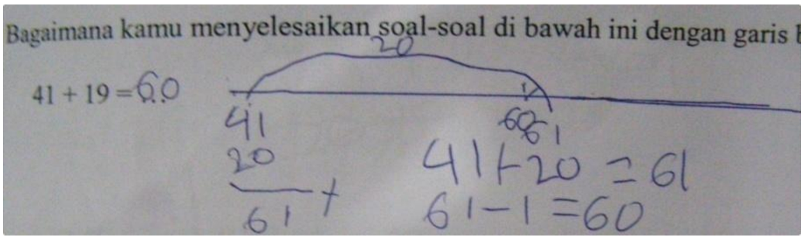

Figure 4 A student becomes flexible, because she sees 19 as 20 minus 1.

In the picture, Yona shows her flexibility in doing addition on an empty number line. She starts writing 41 as the fixed number and doing a jump of 20 at once. Since she knows that 19 is 1 less than 20, she performs another opposite jump to go one step back from 61 to 60. Under the number 41, Yona shows her strategy of how to get 61 from 41 plus 20 by algorithm. It seems useless, because children at Yona's level do not need to perform an algorithm to find the answer of 41 + 20. From an interview and a classroom observation, Yona knows the place value pattern very well, therefore answering 41 plus 20 is an automatisation for her. The next two mathematical sentences (41 + 20 = 61 and 61 -1 = 60) explain how she performs jumps on the empty number line.

The instruction of the problem below is: "*How do you solve this problem on an empty number line?*"

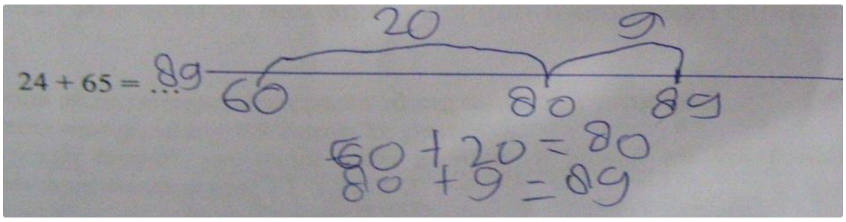

Figure 5 In this respect, the student starts from the second number (65) which is bigger than the first number (24), so she doesn't need to make a big jump for the "60"

Yona probably uses splitting in her mental calculation, i.e. separating the tens and ones, then operating the tens and ones separately, finally adding the tens and the ones to get the result. She is very flexible in this case, because she prefers the splitting strategy for the addition problem which doesn't require conversion of tens. In this case, Yona shows how she represents her thinking on the empty number line by adding 20 to 60. She prefers to put 60, then 20 as the first start, because adding 20 to 60 is much easier than adding 60 to 20. Then she adds the 9 (she put 4 and 5 together) to go to 89 from 80. Actually, the splitting strategy has never been discussed in the classroom; therefore it is a surprise that Yona can come up with the splitting strategy on an empty number line.

CONCLUSION

The result shows that the measuring activity as the contextual situation in the learning trajectory could enhance students' strategies in counting, i.e. from counting one by one to counting by twos, by fives, or by tens. Thus, students consider that the structure of the beads with alternating colors could foster their counting while measuring. Later on, students explore number relations on an empty number line as the basis for performing mental arithmetic strategies. By exploring number relations through number line, students become flexible in doing mental arithmetic strategies. The empty number line can be used not only to represent students' strategies, but also to correct mistakes. Students' contribution play a significant role in developing the mathematical thinking of the classroom community.

However, there are some suggestions that should be emphasized in the teaching practice. First, avoid instructions that suggest students to think that an empty number line is a newly taught procedure in doing calculations, such as giving direct instruction to use an empty number line. Second, give more opportunities for students to share and discuss their strategies verbally without having them write the strategies. In this respect, we should focus on "How to solve the problem mentally?" instead of "How do you solve the problem using an empty number line?". Furthermore, exploring number relations in which a number is decomposed (for instance, 28 = 20 + 8 or 10 + 10 + 8) could help children understand the mental process involved when dealing with the empty number line. Finally, an empty number line is no longer used when students are able to perform mental arithmetic strategies flexibly.

REFERENCES

Beishuizen, Meindert. July 1993. 'Mental Strategies and Materials or models for Addition and Subtraction Up to 100 in Dutch Second Grades', *Journal for Research in Mathematics Education*, Vol.24, No.4, pp. 294 – 323

Cobb, P; Yackel, E. 1996. 'Constructivist, Emergent, and Socio-cultural Perspectives in the Context of Developmental Research', *Educational Psychologist,* 31(3/4), 175 - 190

Cobb, P; Stephan, M, McClain, K; Gravemeijer, K. 2001. 'Participating in Classroom Mathematical Practice', *The Journal of the Learning Science,* 10 (1 & 2), 113 – 163

Cobb, Paul; Confrey, Jere; diSessa, Andrea; Lehrer, Richard. Jan – Feb 2003. 'Design Experiments in Educational Research', *Educational Researcher*, Vol.32. No.1, pp.9 – 13

Freudenthal, Hans. 1991. *Revisiting Mathematics Education: China Lectures.* Dordrecht, the Netherlands: Kluwer Academic Publisher

Gravemeijer, Koeno. 1994a. 'Educational Development and Educational Research in Mathematics Education', *Journal for Research in Mathematics Education* 25: 443-71.

Gravemeijer, Koeno. 1994b. *Developing Realistic Mathematics Education.* Utrecht: Cd-ß Press

Gravemeijer, Koeno; Cobb, Paul. 2006. 'Design Research from a Learning Design Perspective', *Educational Design Research*. London and New York: Routledge, pp.17-51

Klein, A.S. 1998. *Flexibilization of Mental Arithmetic Strategies on a Different Knowledge Base: The Empty Number Line in a Realistic versus Gradual Program Design.* Leiden: Grafisch bedrijf UFB

Reys, R.E; Reys, B.J.; Nohda, N; Emori, H. 1995. 'Mental Computation Performance and Strategy Use of Japanese Students in Grades 2, 4, 6, and 8', *Journal for Research in Mathematics education.* July 1995, Vol.26, No. 4: 304 – 326

Sembiring, R.K.; Hadi, S; Dolk, M. 2008. 'Reforming mathematics learning in Indonesia classrooms through RME', *ZDM Mathematics Education,* DOI 10.1007/s11858-008-0125-9

Treffers, Adrian. 1987. *Three Dimensions (A Model of Goal and Theory Description in Mathematics Instruction – The Wiskobas Project).* Dordrecht, Boston, Lancaster, Tokyo: D.Reidel Publishing Company

Treffers, Adrian. 1991. 'Meeting Innumeracy at Primary School', *Educational Studies in Mathematics ,* 22, pp.333-352

Van den Heuvel-Panhuizen, M. 2001. 'Mental Arithmetic', *Student Learn Mathematics (A learning-teaching trajectory with intermediate attainment targets for calculation with whole numbers in primary school).* The Netherlands: Freudenthal Institute, Utrecht University