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GLOBAL SMOOTH SOLUTIONS TO A CLASS OF ONE-DIMENSIONAL HYPERBOLIC PROBLEMS WITH BOUNDARY DAMPING

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Abstract. In this paper an initial–boundary-value problem for a nonlinear string (or wave) equation with nonclassical boundary conditions is considered. One end of the string is assumed to be fixed and the other end of the string is attached to a dashpot system, where the (positive) damping is generated. Existence, uniqueness, and regularity of solutions to this problem are investigated.

1. Introduction

It is the purpose of this paper to show the global existence and regularity of solutions to the initial–boundary-value problem (P)

$$\begin{aligned} u_{tt} - u_{xx} &= cu_t - \sigma(u_t), \quad 0 < x < \pi, \quad t > 0, && (1.1)\\ u(0,t) &= 0, \quad t \geq 0, && (1.2)\\ u_x(\pi,t) &= -\alpha u_t(\pi,t), \quad t \geq 0, && (1.3)\\ u(x,0) &= u_0(x), \quad 0 \leq x \leq \pi, && (1.4)\\ u_t(x,0) &= u_1(x), \quad 0 \leq x \leq \pi, && (1.5)\end{aligned}$$

where c and α are positive constants, where σ is a monotonic, increasing, and continuous function with $\sigma(0) = 0$, and where u_0 and u_1 satisfy certain regularity conditions, which will be given later. In this initial–boundary-value problem the function u describes the vertical displacement of a nonlinear string which is fixed at $x = 0$ and which is attached at $x = \pi$ to a dashpot system which generates positive damping. These problems can be applied to galloping oscillations of overhead transmission lines; see for instance [6]. For the case of a Dirichlet boundary condition at both boundaries the well-posedness of the weak solution can be found in [3], even for higher

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dimensions. For the case $\sigma(u_t) = \epsilon u_t^3$ with Dirichlet boundary conditions, the problem has been studied in [1] and [6]. In [7] the initial–boundary-value problem (P)(1.1)–(1.5) has also been studied numerically for $\sigma(u_t) = Cu_t^3$, C a constant.

Our goal is to establish the global existence of solutions to (P)(1.1)–(1.5) which are $C^2([0,\pi] \times [0,\infty))$. One verifies that the functions u_0 and u_1 also satisfy the following compatibility conditions:

$$u_0(0) = 0, \quad u_0''(0) = 0, \quad u_1(0) = 0, \tag{1.6}$$

$$u_0'(\pi) + \alpha u_1(\pi) = 0, \quad u_1'(\pi) + \alpha(u_0''(\pi) + cu_1(\pi) - \sigma(u_1(\pi))) = 0. \tag{1.7}$$

Before stating our main result we make some comments about uniqueness and "continuous dependence" upon the data. Suppose u and $\tilde{u}$ are two solutions of the initial–boundary-value problem (P)(1.1)–(1.5). Let $w(x,t) = u(x,t) - \tilde{u}(x,t)$ for $(x,t) \in [0,\pi] \times [0,T]$. By subtracting the PDE's for u and $\bar{u}$ from each other, then by multiplying by w_t, and then by integrating with respect to t from 0 to t and with respect to x from 0 to π, and by making use of the fact that σ is increasing, we obtain the following energy estimate:

$$E_k(t) + E_p(t) + \alpha \int_0^t w_t^2(\pi, s)ds \leq E_k(0) + E_p(0) + 2c \int_0^t E_k(s)ds, \tag{1.8}$$

where

$$E_k(t) = \frac{1}{2} \int_0^\pi w_t^2(x,t)dx \quad \text{and} \quad E_p(t) = \frac{1}{2} \int_0^\pi w_x^2(x,t)dx.$$

Using Gronwall's inequality it follows from (1.8) that

$$E_k(t) + E_p(t) \leq (E_k(0) + E_p(0))e^{2ct}. \tag{1.9}$$

Now given $u(x,0) = \tilde{u}(x,0)$ and $u_t(x,0) = \tilde{u}_t(x,0)$ for $0 \leq x \leq \pi$ it can readily be seen from (1.9) that if a solution exists then it is unique ($w(x,t) = 0 \Longrightarrow u_1(x,t) = u_2(x,t)$). It also follows from (1.9) that if the solution exists it "depends continuously" on the initial values. Now we are in a position to state our main result. It is formulated in the following theorem.

Theorem 1.1. *Suppose $\sigma \in C^2(\mathbb{R})$, $\sigma' \geq 0$, and σ satisfies $\sigma(0) = 0$. Let α and c be positive constants. Suppose $u_0 \in C^3[0,\pi]$ and $u_1 \in C^2[0,\pi]$ satisfy the conditions* (1.2)–(1.3) *and* (1.6)–(1.7). *Then problem* (P)(1.1)–(1.5) *has a unique, twice continuously differentiable solution for $(x,t) \in [0,\pi] \times [0,T]$, where T is an arbitrary positive constant.*

Remark 1.1. It will follow from our proof that the conditions on u_0 and u_1 can be weakened to $u_0 \in H^3(0,\pi)$ and $u_1 \in H^2(0,\pi)$.

This paper is organized as follows. In Section 2 we show that problem (P)(1.1)–(1.5) can be rewritten as a differential equation in an appropriate Hilbert space $\mathcal{H}$,

$$\frac{dz}{dt}(t) = Az(t) + Bz(t), \quad z(0) = z_0. \tag{1.10}$$

Here A is a nonlinear, m-dissipative operator in $\mathcal{H}$ and B is a linear, bounded operator in $\mathcal{H}$. The existence of global solutions follows from a theorem of Kato [5]. This approach requires only that σ be continuous and provides a unique solution

$$u \in W^{2,\infty}((0,T); L^2(0,\pi)) \cap W^{1,\infty}((0,T); H^1(0,\pi)) \cap L^\infty((0,T); H^2(0,\pi)). \tag{1.11}$$

Assuming $\sigma \in C^1(\mathbb{R})$ we show by using a regularity result for linear equations in $\mathcal{H}$ that the solution

$$u \in C^2([0,T]; L^2(0,\pi)) \cap C^1([0,T]; H^1(0,\pi)) \cap C([0,T]; H^2(0,\pi)). \tag{1.12}$$

In Section 3 we establish C^2 regularity by writing an integral equation for u_t. This allows us to establish the required regularity under the assumptions of Theorem 1.1.

Notation. In this paper we will use the following notation. L^2, H^1, and H^2 stand for $L^2(0,\pi)$, $H^1(0,\pi)$, and $H^2(0,\pi)$ respectively. Similarly, C, C^1, and C^2 stand for $C[0,\pi]$, $C^1[0,\pi]$, and $C^2[0,\pi]$ respectively. Also, we will use the somewhat uncommon notation H_0^1 for $\{u \in H^1; u(0) = 0\}$.

2. Global strong solutions

It will be shown that the initial–boundary-value problem (P)(1.1)–(1.5) possesses a unique "strong" global solution. To show the well-posedness of the problem we rewrite problem (P)(1.1)–(1.5) as a system in an appropriate Hilbert space. Setting $v(t) = u(\cdot,t)$, $w(t) = u_t(\cdot,t)$, and $z(t) := (v(t), w(t))^T$. We obtain an equation of the form

$$\frac{dz}{dt}(t) = Az(t) + Bz(t), \quad t \geq 0, \quad z(0) = z_0, \tag{2.1}$$

where

$$Az(t) := A\begin{pmatrix} v(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} w(t) \\ v(t)_{xx} - \sigma(w(t)) \end{pmatrix}, \tag{2.2}$$

and

$$Bz(t) := B\begin{pmatrix} v(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} 0 \\ cw(t) \end{pmatrix}, \tag{2.3}$$

and where $z(0) = (u_0, u_1)^T$. This motivates the following definitions. Let $\mathcal{H} := \{z = (v, w) \in H_0^1 \times L^2\}$ be equipped with the inner product

$$\langle z, \tilde{z} \rangle = \int_0^\pi (v_x \tilde{v}_x + w\tilde{w})\,\mathrm{dx} \tag{2.4}$$

$$= \langle v, \tilde{v} \rangle_1 + \langle w, \tilde{w} \rangle_2. \tag{2.5}$$

Then $\mathcal{H}$ is a Hilbert space. Let

$$D(A) := \{z = (v, w) \in H^2 \cap H_0^1 \times H_0^1; v_x(\pi) + \alpha w(\pi) = 0\}, \tag{2.6}$$

and A be as in (2.2). It should be observed that the operator B defined in (2.3) is linear and bounded in $\mathcal{H}$. To show the global solvability of (2.1) we apply the following theorem.

Theorem 2.1. *Let A be a (single-valued) m-dissipative operator in the Hilbert space $\mathcal{H}$, and let B be a Lipschitz operator in $\mathcal{H}$. For every $z_0 \in D(A)$ and every $T > 0$ there is a unique, absolutely continuous $z : [0, T] \to \mathcal{H}$ such that $z(0) = z_0$ and (2.1) holds at almost every $t > 0$. Also, z is Lipschitz continuous and right-differentiable with $z(t) \in D(A)$ for all $t > 0$ and $\frac{d^+z}{dt} = Az(t) + Bz(t)$, for all $t \geq 0$.*

This theorem is a slight modification of a theorem of Kato (see [5]). Let $\omega \geq 0$ be the Lipschitz constant of the operator B. It is easy to see that $B - \omega I$, where I is an identity operator, is dissipative and Lipschitz continuous, and therefore it is m-dissipative. From the m-dissipativity of A and $B - \omega I$ it follows that $A + B - \omega I$ is m-dissipative. To complete the proof we now can apply the theorem of Kato which can be found for instance in [5, page 180]. Moreover, if $z : [0, T] \longrightarrow E$ is the solution of (2.1), then the solution z is Lipschitz continuous and right-differentiable with $z \in D(A)$.

To show the solvability of the abstract Cauchy problem (2.1) according to Theorem 2.1 we need only the following lemma.

Lemma 2.1. *Let the function σ be monotonically increasing and continuous with $\sigma(0) = 0$, and let $\alpha > 0$. Then the nonlinear operator A defined in (2.2) is m-dissipative on $\mathcal{H}$, and $D(A)$ is dense in $\mathcal{H}$.*

Proof. First we show that $D(A)$ is dense in $\mathcal{H}$. Let us introduce the subspaces $E := \{v \in H^2; v(0) = v_x(\pi) = 0\}$ and $F := \{w \in H^1; w(0) = w(\pi) = 0\}$. It is standard that E is dense in H_0^1 and F is dense in L^2. Moreover $E \times F \subset D(A)$. Hence $D(A)$ is dense in $\mathcal{H}$.

To complete the proof of this lemma we have to show that A is dissipative, and that the range of $\lambda I - A$ is equal to $\mathcal{H}$ for a $\lambda > 0$. Now we show that A

is a dissipative operator. Let $z, \tilde{z} \in D(A)$. A straightforward computation shows that (using the fact that σ is monotonic increasing)

$$\begin{aligned}
&\langle Az - A\tilde{z}, z - \tilde{z}\rangle \\
&= \int_0^\pi \left[(w-\tilde{w})_x\,(v-\tilde{v})_x + \left((v-\tilde{v})_{xx} + \sigma(\tilde{w}) - \sigma(w)\right)(w-\tilde{w})\right] dx \\
&= -\alpha(w(\pi) - \tilde{w}(\pi))^2 - \int_0^\pi (\sigma(w) - \sigma(\tilde{w}))(w - \tilde{w})dx \le 0. \qquad (2.7)
\end{aligned}$$

So we have shown that the nonlinear operator A is a dissipative operator. Secondly, for any $z_0 \in \mathcal{H}$ with $z_0 = (g, h)$ we will show that there exists a $z \in D(A)$ such that

$$(I - A)z = z_0, \qquad (2.8)$$

or equivalently

$$\begin{aligned}
v &= w + g, && (2.9)\\
w &= v_{xx} - \sigma(w) + h, && (2.10)\\
v(0) &= 0, \quad v_x(\pi) + \alpha w(\pi) = 0. && (2.11)
\end{aligned}$$

Let us assume that $g \in H^2 \cap H_0^1$. Then it follows that

$$\begin{aligned}
y &= y_{xx} - \sigma(y) + f, && (2.12)\\
y(0) &= 0, \quad y_x(\pi) + \alpha y(\pi) = -g_x(\pi), && (2.13)
\end{aligned}$$

where $f = h + g_{xx} \in L^2$ and where $y = v - g$. To show that the boundary-value problem (2.12)–(2.13) is solvable we will apply a variational method by introducing the functionals $\langle , \rangle$, φ, and J from H_0^1 into R which are defined by

$$\begin{aligned}
\langle y, y\rangle &= \int_0^\pi \left((y')^2 + y^2\right) dx, \quad \varphi(y) = \int_0^\pi f y\, dx, && (2.14)\\
J(y) &= \int_0^\pi j(y)dx + \frac{\alpha}{2}\Big(y(\pi) + \frac{1}{\alpha} g_x(\pi)\Big)^2, && (2.15)
\end{aligned}$$

where $j(s) = \int_0^s \sigma(\xi)d\xi$ with $s \in \mathbb{R}$. For $y \in H_0^1$ we define the functional $I(\cdot)$ by

$$I(y) = \frac{1}{2}\langle y, y\rangle - \varphi(y) + J(y). \qquad (2.16)$$

It is clear that the functional I is continuous. From the fact that σ is monotonic increasing it follows that the functional I is coercive. Next, to see that I is convex it enough to show that the functional j is convex. Let $a, b \in \mathbb{R}$ with $a \le b$. For any $\lambda \in (0, 1)$ it is easy to see that $a < (1 - \lambda)a + \lambda b < b$. By using the mean-value theorem and the fact that σ is monotonic increasing it follows that $j((1-\lambda)a+\lambda b) \leq (1-\lambda)j(a)+\lambda j(b)$. From convexity, coercivity, and continuity of I it follows that there exists a unique $\bar{y} \in H_0^1$ such that $I(\bar{y}) \leq I(y)$ for all $y \in H_0^1$. Now for arbitrary $y \in H_0^1$ we define $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\phi(t) := I(\bar{y}+ty). \tag{2.17}$$

Since ϕ is continuously differentiable and $\phi(0)$ is minimal it follows that $\bar{y} \in H^2$ and satisfies

$$\int_0^\pi \big(\bar{y}-\bar{y}''+\sigma(\bar{y})-f\big)\, y\, dx + \big(\bar{y}'(\pi)+\alpha\bar{y}+g_x(\pi)\big)\, y(\pi) = 0. \tag{2.18}$$

We have to notice that the equation (2.18) holds for every $y \in H_0^1$. So $\bar{y}$ is the solution of the boundary-value problem (2.12)–(2.13) in the sense of distributions.

Now let us assume that $g \in H_0^1$. Then there exists a sequence g_n in $H^2 \cap H_0^1$ such that $g_n \longrightarrow g$ in H^1. For all $n \in \mathbb{N}^+$ there is a unique $z_n = (v_n, w_n) \in D(A)$ such that

$$\begin{aligned} v_n &= w_n + g_n, && (2.19)\\ w_n &= v_{n_{xx}} - \sigma(w_n) + h, && (2.20)\\ v_n(0) &= 0, \quad v_{n_x}(\pi) + \alpha w_n(\pi) = 0. && (2.21) \end{aligned}$$

Since A is a dissipative operator on $\mathcal{H}$ an a-priori estimate can be obtained, that is,

$$\|z_n - z_m\|_{\mathcal{H}} \leq \|f_n - f_m\|_{\mathcal{H}}, \tag{2.22}$$

where $f_n = (g_n, h)^T \in \mathcal{H}$. From (2.22) it follows that $\{v_n\}$ and $\{w_n\}$ are Cauchy sequences in H^1 and L^2 respectively. Moreover, $(H^1, \langle\ ,\ \rangle_1)$ and $(L^2, \langle\ ,\ \rangle_2)$ are complete, implying that there are $\bar{v} \in H^1$ and $\bar{w} \in L^2$ such that $v_n \longrightarrow \bar{v}$ and $w_n \longrightarrow \bar{w}$. Furthermore, $\{v_n\}$ is also a Cauchy sequence in C^0 with maximum norm. Therefore we obtain $\bar{v} \in H_0^1$. From the continuity of σ it follows from (2.19) and (2.20) for $n \longrightarrow \infty$ that

$$w_n = v_n - g_n \longrightarrow \bar{v} - g = \bar{w} \in H_0^1, \tag{2.23}$$

and

$$v_{n_{xx}} = w_n + \sigma(w_n) - h \longrightarrow \bar{w} + \sigma(\bar{w}) - h \in L^2. \tag{2.24}$$

Next we will show that $\bar{v} \in H^2 \cap H_0^1$ and that $v_{n_{xx}}$ converges to $\bar{v}_{xx}$. Since v_n is in H^2 it follows that there are positive constants c_1 and c_2 such that

$$\|(v_n - v_m)'\|_{L^2} \leq c_1\|v_n - v_m\|_{L^2} + c_2\|(v_n - v_m)''\|_{L^2}. \tag{2.25}$$

It can readily be seen from (2.25) that $v_n' \longrightarrow \bar{v}'$ in L^2. For $n \longrightarrow \infty$ it follows from

$$\int_0^\pi v_n'\varphi' dx = -\int_0^\pi v_n''\varphi\, dx$$

(for arbitrary $\varphi \in H^2$ where φ vanishes for $x = 0$ and π) that

$$\int_0^\pi \bar{v}'\varphi' dx = -\int_0^\pi (\bar{w} + \sigma(\bar{w}) - h)\varphi\, dx.$$

So, $\bar{v} \in H^2$, and $\bar{v}'' = \bar{w} + \sigma(\bar{w}) - h$ in L^2 with $\bar{v}(0) = 0$. It also follows from (2.24) that $v_{n_{xx}} \longrightarrow \bar{v}''$ in L^2. Finally, we have to show that $\bar{v}$ and $\bar{w}$ satisfy the boundary conditions (2.11). To show this we integrate (2.20) once, yielding

$$v_{n_x}(x) = -\int_x^\pi (w_n + \sigma(w_n) - h)dx - \alpha w_n(\pi). \tag{2.26}$$

It can also be shown that $w_n \longrightarrow \bar{w}$ uniformly in C with the maximum norm. Again by using the continuity of σ and the Lebesgue monotone convergence theorem it follows from (2.26) that

$$\bar{v}_x(x) = -\int_x^\pi (\bar{w} + \sigma(\bar{w}) - h)dx - \alpha\bar{w}(\pi). \tag{2.27}$$

We deduce from (2.23) and (2.27) that for any $z_0 \in \mathcal{H}$ there is $z \in D(A)$ such that the equations (2.9)–(2.11) hold. This completes the proof of the lemma. $\square$

Let $z_0 = \binom{u_0}{u_1} \in D(A)$, and let $z : [0, \infty) \to \mathcal{H}$ be the corresponding solution to problem (2.1) given by the Theorem 2.1. Set

$$z(t) = \begin{pmatrix} v(t) \\ w(t) \end{pmatrix},\ t \geq 0. \tag{2.28}$$

Then we have for every $T > 0$,

$$v \in W^{1,\infty}((0,T); H_0^1) \tag{2.29}$$

$$w \in W^{1,\infty}((0,T); L^2) \tag{2.30}$$

$$v(t) \in H^2 \cap H_0^1, \quad \text{for every} \quad t \geq 0 \tag{2.31}$$

$$w(t) \in H_0^1, \quad \text{for every} \quad t \geq 0 \tag{2.32}$$

$$\frac{d^+v}{dt}(t) = w(t) \quad \text{in} \quad H^1 \quad \text{for every} \quad t \in [0,T] \tag{2.33}$$

$$\frac{d^+w}{dt}(t) = (v(t))_{xx} + cw(t) - \sigma(w(t)) \text{ in } L^2 \text{ for every } t \in [0,T]. \tag{2.34}$$

Notice that since $w(t) \in H_0^1$ $(w(t) \in C)$, hence $\sigma(w(t))(x) := \sigma(w(x,t))$ is well-defined for $x \in [0,\pi]$, and $\sigma(w(t)) \in C$. Next we want to establish that the function z satisfies the following integral equation in $\mathcal{H}$:

$$z(t) = z_0 + A_0 \int_0^t z(s)ds + \int_0^t f(s)ds, \quad t \geq 0, \tag{2.35}$$

where the operator A_0 is defined as the operator A but with $\sigma(s) = cs$, $s \in \mathbb{R}$, and where

$$f(t) := \begin{pmatrix} 0 \\ cw(s) - \sigma w(s) \end{pmatrix}, \quad t \geq 0. \tag{2.36}$$

Notice that the operator A_0 is *linear* m-dissipative and densely defined. If we can show that the function $f \in W^{1,1}((0,T);\mathcal{H})$, then it will follow from Theorem 8.1 in [2] that $z \in C^1([0,T];\mathcal{H}) \cap C([0,T];D(A_0))$ for every $T > 0$. From this we shall obtain $v \in C^2([0,T];L^2) \cap C^1([0,T];H_0^1) \cap C([0,T];H^2)$. In order to prove (2.35) we use the fact that

$$w \in C([0,T];C) \quad \text{for every} \quad T > 0. \tag{2.37}$$

Indeed, it follows from (2.30) that $w \in W^{1,\infty}((0,T);L^2)$ and from (2.29) and (2.33) that $w \in L^\infty((0,T);H_0^1)$. Then (2.37) follows from Lemma $A1$ from the appendix. Since $v \in W^{1,\infty}((0,T);H_0^1)$, $w \in L^\infty((0,T);H_0^1)$, and $\frac{dv}{dt} = w$ almost everywhere in H_0^1 we have

$$v(t) = u_0 + \int_0^t w(s)ds, \quad t \geq 0, \tag{2.38}$$

in H_0^1. From (2.37) we also have $t \to \sigma(w(t)) \in C([0,T];C)$; hence, $cw - \sigma(w) \in C([0,T];L^2)$, and the function f defined in (2.36) belongs to $C([0,T];\mathcal{H})$. From (2.31) and (2.34) we deduce $t \to (v(t))_{xx}$ belongs to $L^\infty((0,T);L^2)$; hence, by integrating in L^2 we get

$$w(t) = u_1 + \int_0^t (v(s))_{xx}ds + \int_0^t (cw(s) - \sigma(w(s)))ds, \quad \forall t \geq 0, \quad \text{in} \quad L^2. \tag{2.39}$$

From (2.30) and (2.33) we have $z(t) \in D(A_0) = D(A)$ and $A_0 z(t) = \binom{0}{w(t)-(v(t)_{xx}}$ for every $t \geq 0$. Since A_0 is linear m-dissipative, densely defined $f \in C([0,T];\mathcal{H})$, it follows from Theorem 6.2 and 7.2 of [2] that there exists a function $\tilde{z} \in C([0,T];\mathcal{H})$ satisfying $\int_0^t \tilde{z}(s)ds \in D(A_0)$ for every $t \in [0,T]$ and

$$\tilde{z}(t) = z_0 + A_0 \int_0^t \tilde{z}(s)ds + \int_0^t f(s)ds, \quad 0 \leq t \leq T. \tag{2.40}$$

Consequently, the function $z - \tilde{z} \in C([0,T];\mathcal{H})$ satisfies

$$z(t) - \tilde{z}(t) = \int_0^t A_0 z(s)ds - A_0 \int_0^t \tilde{z}(s)ds \tag{2.41}$$
$$= \int_0^t (A_0 - I)z(s)ds - (A_0 - I)\int_0^t \tilde{z}(s)ds + \int_0^t (z(s) - \tilde{z}(s))ds.$$

Note that $0 \in \rho(A_0 - I)$ in $\mathcal{H}$; hence,

$$(A_0 - I)^{-1}(z(t) - \tilde{z}(t)) = \int_0^t z(s)ds - \int_0^t \tilde{z}(s)ds + (A_0 - I)^{-1}\int_0^t (z(s) - \tilde{z}(s))ds. \tag{2.42}$$

This shows that $\int_0^t (z(s) - \tilde{z}(s))ds \in D(A_0)$ and

$$(z - \tilde{z})(t) = (A_0 - I)\int_0^t (z(s) - \tilde{z}(s))ds + \int_0^t (z(s) - \tilde{z}(s))ds$$
$$= A_0 \int_0^t (z(s) - \tilde{z}(s))ds. \tag{2.43}$$

Since $z - \tilde{z} \in C([0,T];\mathcal{H})$ and $\int_0^t (z(s) - \tilde{z}(s))ds \in D(A_0)$ it is an integral solution, and by uniqueness (Theorem 6.1 of [2]), $z - \tilde{z} = 0$, hence $z = \tilde{z}$, which establishes (2.35). Next we apply Theorem 8.1 of [2] to show that $z \in C^1([0,T];\mathcal{H}) \cap C([0,T];D(A_0))$. Since $z_0 \in D(A_0)$, it suffices to establish $f \in W^{1,1}((0,T);\mathcal{H})$; i.e., $cw - \sigma(w) \in W^{1,1}((0,T);L^2)$. Note that $w \in W^{1,\infty}((0,T);L^2)$ by (2.30). Since L^2 is a Hilbert space it is sufficient to prove $\sigma(w) \in Lip((0,T);L^2)$. From (2.37) we can define $\tilde{w}(x,t) = w(t)(x)$, $(x,t) \in Q_T := [0,\pi] \times [0,T]$. Then $\tilde{w} \in C(Q_T)$. Set $C_1 := \max_{Q_T} |\tilde{w}|$ and $C_2 := \max_{|\xi| \le C_1} |\sigma'(\xi)|$, where we assume from now on that $\sigma \in C^1(\mathbb{R})$. Then for $0 \le t_1 \le t_2 \le T$, $x \in [0,\pi]$,

$$|\sigma(w(t_1))(x) - \sigma(w(t_2))(x)| = |\sigma(\tilde{w}(x,t_1)) - \sigma(\tilde{w}(x,t_2))|$$
$$\le C_2|\tilde{w}(x,t_1) - \tilde{w}(x,t_2)| = C_2|w(t_1)(x) - w(t_2)(x)|. \tag{2.44}$$

Hence,

$$\|\sigma(w(t_1)) - \sigma(w(t_2))\|_{L^2}^2 \le C_2^2|t_1 - t_2|^2.$$

It follows from (2.30) that there is $C_3 > 0$ such that

$$\|w(t_1) - w(t_2)\|_{L^2}^2 \le C_3^2|t_1 - t_2|^2. \tag{2.45}$$

As a consequence $\sigma(w) \in Lip((0,T);L^2)$, and we are done. Finally from $z \in C^1([0,T];\mathcal{H}) \cap C([0,T];D(A_0))$ and using (2.33) and (2.34), it follows that $v \in C^2([0,T];L^2) \cap C^1([0,T];H^1) \cap C([0,T];H^2)$. Summarizing, we obtain

Proposition 2.1. *Under the assumptions of Theorem* 1.1 *with* $\sigma \in C^1(\mathbb{R})$ *(instead of* $C^2(\mathbb{R})$*), the functions* (v, w) *defined in* (2.28) *have the following regularity:*

$$v \in C^2([0,T];L^2) \cap C^1([0,T];H_0^1) \cap C([0,T];H^2 \cap H_0^1), \tag{2.46}$$

$$w \in C^1([0,T];L^2) \cap C([0,T];H_0^1). \tag{2.47}$$

3. Classical solutions

Let $T > 0$ and let $z \in C^1([0,T];\mathcal{H}) \cap C([0,T];D(A_0))$ be the solution of (2.35) obtained in Proposition 2.1. In this section we shall prove that under the assumptions of Theorem 1.1 the following holds:

$$\dot{z} \in C^1([0,T];\mathcal{H}) \cap C([0,T];D(A_0)). \tag{3.1}$$

As a consequence $z \in C^2([0,T];\mathcal{H}) \cap C^1([0,T];D(A_0))$. Then

$$v \in C^2([0,T];H_0^1) \cap C^1([0,T];H^2 \cap H_0^1), \quad w \in C^2([0,T];L^2) \cap C^1([0,T];H_0^1).$$

Set $Q_T := [0,\pi] \times [0,T]$ and

$$u(x,t) := v(t)(x), \quad \text{for} \quad (x,t) \in Q_T. \tag{3.2}$$

From $v \in C^2([0,T];H_0^1)$ we have $v \in C^2([0,T];C)$ and $v(t)(0) = 0$, $t \in [0,T]$. Moreover $\dot{v}, \ddot{v} \in C([0,T];C)$; hence,

$$u, u_t, u_{tt} \in C(Q_T) \quad \text{and} \quad u(0,t) = 0, \quad t \in [0,T]. \tag{3.3}$$

From $v \in C^1([0,t];H^2 \cap H_0^1)$ we obtain $\bar{v} \in C([0,T];C^1)$; hence,

$$u_x, u_{xt}, u_{xx} \in C(Q_T). \tag{3.4}$$

It follows from (3.3) and (3.4) that $u \in C^2(Q_T)$ and $u_{xt} = u_{tx}$ on Q_T. Finally, from

$$\frac{dw}{dt}(t) = (v(t))_{xx} + cw(t) - \sigma(w(t)), \quad t \in [0,T] \quad \text{in} \quad L^2 \tag{3.5}$$

and $\dot{w} \in C([0,T];C)$, $w \in C([0,T];C)$, $t \to \sigma(w(t)) \in C([0,T];C)$, we obtain $t \to (v(t))_{xx} \in C([0,T];C)$; hence, by integrating in x we get

$$t \to v(t) \in C([0,T];C^2). \tag{3.6}$$

It follows that

$$u_{xx} \in C(Q(T)); \tag{3.7}$$

hence, $u \in C^2(Q(T))$. Consequently, since $w \in C([0,T];C)$ and $w(t)(x) = \dot{v}(t)(x)$, we have $w(t)(x) = u_t(x,t)$, and from (3.5) we obtain

$$u_{tt}(x,t) = u_{xx}(x,t) + cu_t(x,t) - \sigma(u_t(x,t)), \quad \text{in} \quad Q_T. \tag{3.8}$$

Since $z(0) = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$ and $w(0) = (\dot{v})(0)$, we have $u(x,0) = u_0(x)$ and $u_t(x,0) = u_1(x)$, $x \in [0,\pi]$. Finally, from $z \in C([0,T]; D(A_0))$ we have $(v(t))_x(\pi) + \alpha w(t)(\pi) = 0$, $t \in [0,T]$, and Theorem 1.1. is proved. It remains to establish (3.1). From (2.35) and Proposition 2.1 we have

$$\begin{aligned} &\dot{z}(t) = A_0 z(t) + f(t),\ t \in [0,T] \text{ in } \mathcal{H}, \\ &z(0) = z_0 \in D(A_0). \end{aligned} \tag{3.9}$$

Set $q = \dot{z}$; then $q \in C([0,T]; \mathcal{H})$ and $\int_0^t q(s)ds = z(t) - z(0) \in D(A_0)$, $t \in [0,T]$, and

$$q(t) = A_0 z_0 + A_0 \int_0^t q(s)ds + f(t), \quad t \in [0,T] \quad \text{in} \quad \mathcal{H}, \tag{3.10}$$

where $f(t)$ is given by (2.36),

$$f(t) - f(0) = \begin{pmatrix} 0 \\ (cw(t) - \sigma(w(t))) - (cw(0) - \sigma(w(0))) \end{pmatrix}.$$

In view of Lemmas $A2$ and $A4$,

$$(cw(t) - \sigma(w(t))) - (cw(0) - \sigma(w(0))) = \int_0^t R(s)\dot{w}(s)ds \quad \text{ in } L^2,$$

where $R(s)$ is the multiplication operator in L^2 associated with $\rho(y) = cy - \sigma'(y)$, $\rho \in C^1(\mathbb{R})$. Setting $D(t) \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 \\ R(t)\eta \end{pmatrix}$, $t \in [0,T]$, $\begin{pmatrix} \xi \\ \eta \end{pmatrix} \in \mathcal{H}$, we have as a consequence of the same lemmata that the family $\{D(t)\}_{t\in[0,T]}$ of bounded operators in $\mathcal{L}(\mathcal{H})$ satisfies all assumptions of Lemma A5. Moreover, q satisfies

$$q(t) = q_0 + A_0 \int_0^t q(s)ds + \int_0^t D(s)q(s)ds, \quad t \in [0,T] \text{ in } \mathcal{H}, \tag{3.11}$$

where $q_0 = A_0 z_0 + f(0)$; that is,

$$q_0 = A_0 \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} + \begin{pmatrix} 0 \\ cu_1 - \sigma(u_1) \end{pmatrix} = \begin{pmatrix} u_1 \\ (u_0)_{xx} + cu_1 - \sigma(u_1) \end{pmatrix}. \tag{3.12}$$

We claim that under the assumptions of Theorem 1.1 $q_0 \in D(A_0)$. Indeed, $u_1 \in H^2 \cap H_0^1$ and $u_1 \in C^1$. Then $cu_1 - \sigma(u_1) \in C^1$ and $cu_1(0) - \sigma(u_1(0)) = 0$ since $u_1(0) = 0$ and $\sigma(0) = 0$, so $cu_1 - \sigma(u_1) \in H_0^1$. Moreover, $(u_0)_{xx} \in H^1$ since $u_0 \in H^3$ and $(u_0)_{xx}(0) = 0$ from (1.6). Then $(u_0)_{xx} + cu_1 - \sigma(u_1) \in H_0^1$. It remains to show that $\alpha((u_0)_{xx}(\pi) + cu_1(\pi) - \sigma(u_1(\pi))) + (u_1)_x(\pi) = 0$.

But this is (1.7). Therefore $q_0 \in D(A_0)$. From Lemma A5, it follows that q is the only solution r in $C([0,T];\mathcal{H})$ with $\int_0^t r(s)ds \in D(A_0)$, $t \in [0,T]$. Moreover, since $q_0 \in D(A_0)$, $q \in C^1([0,T];\mathcal{H})$ and $r \in C([0,T];D(A_0))$, which completes the proof of Theorem 1.1.

Appendix A

Lemma A1. *Let $T > 0$; then*

$$W^{1,\infty}((0,T);L^2) \cap L^\infty((0,T);H^1) \subset C([0,T];C).$$

Proof. First we observe that

$$W^{1,\infty}((0,T);L^2) \cap L^\infty((0,T);H^1) \subset B([0,T];H^1).$$

Now let $u \in W^{1,\infty}((0,T);L^2) \cap L^\infty((0,T);H^1)$; then there exists $M > 0$ and $N \subset [0,T]$ with measure zero such that for $t \in [0,T]\backslash N$, $u(t) \in H^1$, and $\|u\|_{H^1} \leq M$. Let $\bar{t} \in N$ since $[0,T]\backslash N$ is dense in $[0,T]$; hence, there exists a sequence $\{t_n\} \in [0,T]/N$ such that $t_n \to \bar{t}$. Then $u(t_n) \to u(\bar{t})$ in L^2. Since H^1 is a Hilbert space there is a subsequence $\{t_{n_k}\}_{k=0}^\infty$ and $z \in H^1$ such that $u(t_{n_k}) \rightharpoonup z$ in H^1. Since H^1 is continuously embedded in L^2 then $u(t_{n_k}) \rightharpoonup z$ in L^2; hence, $z = u(\bar{t}) \in H^1$. Moreover, $\|u(\bar{t})\|_{H^1} \leq \lim_{k\to\infty} \|u(t_{n_k})\|_{H^1} \leq M$. Hence $u \in B([0,T];H^1)$. Finally, we prove $u \in C([0,T];C)$. Let $\bar{t}, t_n \in [0,T]$, $n \in \mathbb{Z}^+$ be such that $t_n \to \bar{t}$ as $n \to \infty$. Then $u(t_n) \to u(\bar{t})$ in L^2, and as before since $\|u(t_n)\|_{H^1} \leq M$, $n \geq 1$, for some $M \geq 0$, there exists a subsequence $\{u(t_{n_k})\}$ such that $u(t_{n_k}) \rightharpoonup u(\bar{t})$ in H^1. Since H^1 is compactly embedded in L^2 we have $u(t_{n_k}) \to u(\bar{t})$ in C. Moreover, for any subsequence of $\{t_n\}_{n\geq 0}$ we can extract a subsequence $u(t_{n_k}) \to u(\bar{t})$ in C. So the whole sequence converges to $u(\bar{t})$ in C, and $u \in C([0,T];C)$. $\square$

Lemma A2. *Let $0 \leq T_1 \leq t_1 < t_2 \leq T_1 \leq T$, $r \in C^1([T_1,T_2];L^2) \cap C([T_1,T_2];H^1) \subseteq C([T_1,T_2];C)$. Let $\rho \in C(\mathbb{R})$ and let $M(t) \in \mathcal{L}(L^2)$, $t \in [T_1,T_2]$ be the multiplication operator defined by*

$$(M(t)\eta)(x) = \rho(r(t)(x))\eta(x), \quad \text{for a.e.} \quad x \in [0,\pi] \quad \text{and every} \quad t \in [T_1,T_2]. \tag{A.1}$$

Then the following holds:

(1) $\|M(t)\|_{\mathcal{L}(L^2)} = \max_{x\in[0,\pi]} |\rho(r(t))(x)| \leq C_1$ *where*

$$C_1 := \max_{t\in[T_1,T_2]} \max_{x\in[0,\pi]} |\rho(r(t)(x)|. \tag{A.2}$$

(2) *For every $\bar{r} \in C^1([T_1,T_2];L^2) \cap C([T_1,T_2];H^1)$, $[T_1,T_2] \ni t \to M(t)\bar{r}(t) \in C([T_1,T_2];L^2)$ and $\|M(t)\bar{r}(t)\|_{L^2} \leq C_1 \max_{s\in[T_1,T_2]} \|\bar{r}(s)\|_{L^2}$. Moreover, let $r_n \in C^1([T_1,T_2];L^2) \cap C([T_1,T_2];H^1), n = 1,2,\ldots$ be*

such that $\lim \|r_n - r\|_{C([T_1,T_2];C)} = 0$. *Let* $\{M_n(t)\}_{n=1}^{\infty}$ *be the corresponding multiplication operators; then*

$$\lim_{n\to\infty} \sup_{t\in[T_1,T_2]} \|M_n(t) - M(t)\|_{\mathcal{L}(L^2)} = 0. \tag{A.3}$$

Lemma A3. *Given* $r \in C^1([T_1,T_2];L^2) \cap C([T_1,T_2];H^1)$ *there exists a sequence* $r_n \in C^1([T_1,T_2];C^1), n = 1, 2, \ldots$ *such that*

$$\|\dot{r_n} - \dot{r}\|_{C([T_1,T_2];L^2)} + \|r_n - r\|_{C([T_1,T_2];H^1)} \to 0,\ n \to \infty. \tag{A.4}$$

Lemma A4. *Under the assumptions of Lemma A2, let* ρ, r, $\bar{r}$, *and* $M(t)$ *be as in Lemma A2 with the additional assumption* $\rho \in C^1(\mathbb{R})$. *Let* $\{P(t)\}_{t\in[T_1,T_2]}$ *be the corresponding multiplication operators in* L^2 *associated with* ρ' (*i.e.,* $P(t)\eta = \rho'(r(t)\eta)$). *Then*

$$[T_1,T_2] \ni t \to \rho(r(t)) \in \mathrm{Lip}((T_1,T_2);L^2), \tag{A.5}$$

$$\rho(r(t_2)) - \rho(r(t_1)) = \int_{t_1}^{t_2} P(s)\dot{r}(s)ds, \quad T_1 \le t_1 < t_2 \le T_2, \quad \text{in} \quad L^2, \tag{A.6}$$

and

$$\|\rho(r(t_2)) - \rho(r(t_1))\|_{L^2} \le |t_2 - t_1| \max_{s\in[T_1,T_2]} \|P(s)\|\|\dot{r}\|_{C([T_1,T_2];L^2)}. \tag{A.7}$$

Moreover, $[T_1,T_2] \ni t \to M(t)\bar{r}(t) \in \mathrm{Lip}((T_1,T_2);L^2)$ *and there exists* C_2 *independent of* T_1, T_2, *and* $\bar{r}$ *such that*

$$\|M(t_2)\bar{r}(t_2) - M(t_1)\bar{r}(t_1)\|_{L^2} \le C_2(\|\dot{\bar{r}}\|_{C([T_1,T_2];L^2)} + \|\bar{r}\|_{C([T_1,T_2];H^1)}. \tag{A.8}$$

Proof. For the proof of Lemma A4, one approximates r and $\bar{r}$ by sequence r_n and $\bar{r}_n$ as in A3. The estimates and formulas can be easily established for r_n and $\bar{r}_n$. The result follows by letting n tend to ∞. □

Lemma A5. *Let* $(H, \langle\ ,\ \rangle, \|\cdot\|)$ *be a Hilbert space and let* $L : D(L) \subset H \to H$ *be a linear, m-dissipative operator in* H. (*Then* $D(L)$ *is automatically dense in* H.) *Let* $r_0 \in H$ *and let* $T > 0$. *Suppose that* $\{D(t)\}_{t\in[0,T]}$ *is a family of bounded, linear operators in* H *such that for every* $0 \le T_1 < T_2 \le T$ *and every* $r \in C([T_1,T_2];H)$,

$$[T_1,T_2] \ni t \to D(t)r(t) \in C([T_1,T_2];H) \tag{A.9}$$

and $\max_{t\in[T_1,T_2]} \|D(t)r(t)\| \le M \max_{t\in[T_1,T_2]} \|r(t)\|$ *for some* M *independent of* $T_1, T_2 \in [0,T]$. *Then*

(1) *there exists a unique* $\bar{r} \in C([0,T];H)$ *such that for every* $t \geq 0$, $\int_0^t \bar{r}(s)ds \in D(L)$ *and*

$$\bar{r}(t) = r_0 + L\int_0^t \bar{r}(s)ds + \int_0^t D(s)\bar{r}(s)ds. \tag{A.10}$$

(2) *Assume moreover* $r_0 \in D(L)$, *and for every* $0 \leq T_1 < T_2 \leq T$, *and for every* $r \in C^1([T_1,T_2];H) \cap C([T_1,T_2];D(L))$, $[T_1,T_2] \ni t \to D(t)r(t) \in \mathrm{Lip}((T_1,T_2);H)$ *and* $\exists M' > 0$ *independent of* T_1 *and* T_2 *such that*

$$\|D(t_1)r(t_1) - D(t_2)r(t_2)\| \leq M'|t_1 - t_2|\|r\|_{[T_1,T_2]} \tag{A.11}$$

for every $0 \leq T_1 \leq t_1 < t_2 \leq T_2 \leq T$, *where*

$$\|r\|_{[T_1,T_2]} := \max_{t\in[T_1,T_2]} \|\dot{r}(t)\| + \max_{t\in[T_1,T_2]} \|(L-I)r(t)\|.$$

Then the function $\bar{r}$ *defined in* (*A*.10) *satisfies* $\bar{r} \in C^1([0,T];H) \cap C([0,T];D(L))$ *and*

$$\frac{d\bar{r}}{dt}(t) = L\bar{r}(t) + D(t)\bar{r}(t), \quad \text{for every} \quad t \in [0,T], \tag{A.12}$$

$$\bar{r}(0) = r_0. \tag{A.13}$$

Proof. Lemma A5 follows from a standard application of the contraction principle in the space $C([0,T];H)$ and $C^1([T_1,T_2];H) \cap C([T_1,T_2];D(L))$ and from Theorems 6.1, 8.1, and Corollary 7.3 of [2]. □

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