On Boundary Damping for a Weakly Nonlinear Wave Equation

Darmawijoyo* and W.T. van Horssen†

Abstract

In this paper an initial-boundary value problem for a weakly nonlinear string (or wave) equation with non-classical boundary conditions is considered. One end of the string is assumed to be fixed and the other end of the string is attached to a spring-mass-dashpot system, where the damping generated by the dashpot is assumed to be small. This problem can be regarded as a rather simple model describing oscillations of flexible structures such as suspension bridges or overhead transmission lines in a wind field. A multiple time-scales perturbation method will be used to construct formal asymptotic approximations of the solution. It will also be shown that all solutions tend to zero for a sufficiently large value of the damping parameter.

Keywords: wave equation, boundary damping, asymptotics, two-timescales perturbation method.

1 Introduction

There are a number of examples of flexible structures such as suspension bridges, overhead transmission lines, dynamically loaded helical springs that are subjected to oscillations due to different causes. Simple models which describe these oscillations can be expressed in initial-boundary value problems for wave equations like in [2], [5], [6], [10], [12], [13] or for beam equations like in [3], [4], [14], [15]. Simple models which describe these oscillations can involve linear or nonlinear second and fourth order partial differential equations with classical or non-classical boundary conditions. These problems have been studied in [2],

*On leave from State University of Sriwijaya, Indonesia

†Department of Applied Mathematical Analysis, Faculty of Information Technology and Systems, Delft University of Technology, Mekelweg 4, 2628 CD Delft, The Netherlands, Email: Darmawijoyo@its.tudelft.nl and W.T.vanHorssen@its.tudelft.nl

[3],[4],[5], [6], [10], [12], [13] using a two-timescales perturbation method or a Galerkin-averaging method to construct approximations.

In most cases simple, classical boundary conditions are applied (such as in [2], [3], [4], [10], [12], [13], [14]) to construct approximations of the oscillations. More complicated, non-classical boundary conditions (see for instance [5], [6], [11], [15], [16], [17], [18]) have been considered only for linear partial differential equations. In this paper we will study an initial-boundary value problem for a weakly nonlinear partial differential equation for which one of the boundary conditions is of non-classical type. Asymptotic approximations of the solution will be constructed. In fact, we will consider the vibrations of a string which is fixed at $x = 0$ and is attached to a spring-mass-dashpot system at $x = \pi$ (see also figure 1). This problem can be considered as a rather simple model to describe wind-induced vibrations of an overhead transmission line or a bridge (see [3,13]).

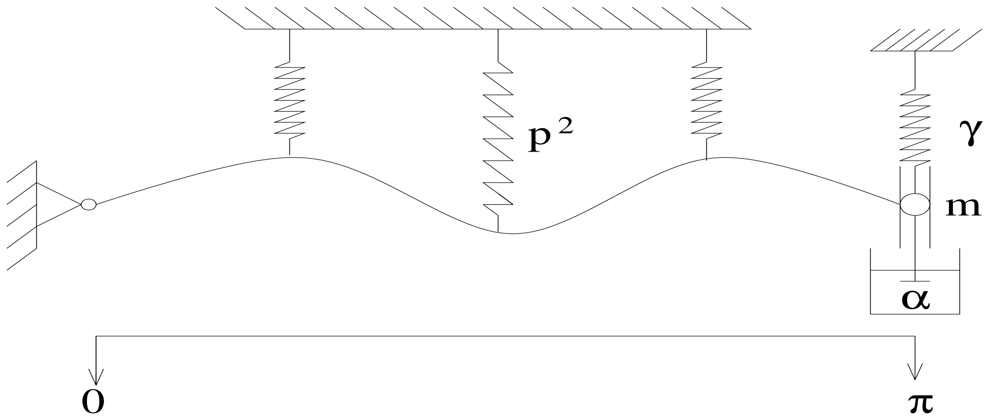

Figure 1: A Simple model of a suspension bridge.

It is assumed that ρ (the mass-density of the string), T (the tension in the string), $\tilde{m}$ (the mass in the spring-mass-dashpot system), $\tilde{\gamma}$ (the stiffness of the spring), $\tilde{\epsilon}$ (the damping coefficient of the dashpot with $0 < \tilde{\epsilon} \ll 1$), and p^2 (for instance the spring constant of the the stays of the bridge) are all positive constants. Moreover, we only consider the vertical displacement $\tilde{u}(x, \tilde{t})$ of the string, where x is the place along the string, and $\tilde{t}$ is time. We neglect internal damping and consider the weight W of the string per unit length to be constant ($W = \mu g$, g is the gravitational acceleration). We consider a uniform wind flow, which causes nonlinear drag and lift forces (F_D, F_L) to act on the structure per unit length. After some scalings, the equation describing the vertical displacement of the string is:

$$\bar{u}_{tt} - \bar{u}_{xx} + p^2\bar{u} = -\mu g + F_D + F_L. \tag{1.1}$$

After some calculations (see also [13]) the following initial-boundary value problem is obtained as simple model to describe the wind-induced oscillations of the string(or bridge)

$$u_{tt} - u_{xx} + p^2 u = \epsilon\left(u_t - \frac{1}{3}u_t^3\right), \ 0 < x < \pi, \ t > 0, \tag{1.2}$$

$$
\begin{aligned}
u(0,t) &= 0,\ t \geq 0, && (1.3)\\
u_x(\pi,t) &= -\epsilon\left(mu_{tt}(\pi,t)+\gamma u(\pi,t)+\alpha u_t(\pi,t)\right),\ t\geq 0, && (1.4)\\
u(x,0) &= \phi(x),\ 0<x<\pi, && (1.5)\\
u_t(x,0) &= \psi(x),\ 0<x<\pi, && (1.6)
\end{aligned}
$$

where ϕ and ψ are the initial displacement and the initial velocity of the string respectively, and where p^2, m, γ, and α are positive constants, and where $0 < \epsilon \ll 1$. In this paper formal approximations (that is, functions that satisfy the differential equation and the initial and boundary values up to some order in ϵ) will be constructed for the initial-boundary value problem (1.2) - (1.6).

The outline of this paper is as follows. In section 2 we apply a two-timescales perturbation method to construct formal approximations for the solution of the initial-boundary value problem (1.2) - (1.6) and we analyze this solution. Also in section 2 we show that for all values of p^2 mode interactions occur only between modes with non-zero initial energy (up to $O(\epsilon)$). Moreover, it will be shown in section 2 that for $\alpha \geq \frac{\pi}{2}$ all solutions tend to zero. In section 3 we make some remarks and draw some conclusions.

2 The construction of asymptotic approximations

To construct formal asymptotic approximations for the solution of the initial-boundary value problem (1.2) - (1.6) a two-timescales perturbation method (see [1,7,8]) will be used in this section. Since an approximation in the form of an infinite series will be constructed we will impose some additional conditions on the initial values in order to get a convergent series representation for which summation and differentiation may be interchanged. The additional conditions on the initial values are: $\phi(0)=\phi'(\pi)=\phi''(0)=\phi'''(\pi)=\psi(0)=\psi'(\pi)=\psi''(0)=0, \phi\in C^4([0,\pi],\Re),\ \psi\in C^3([0,\pi],\Re)$. By using a two-timescales perturbation method the function $u(x,t)$ is supposed to be a function of x, t, and τ, where $\tau=\epsilon t$. We put

$$u(x,t)=v(x,t,\tau;\epsilon). \tag{2.1}$$

By substituting (2.1) into the initial-boundary value problem (1.2) - (1.6) we obtain

$$
\begin{aligned}
v_{tt}-v_{xx}+p^2v+2\epsilon v_{t\tau}+\epsilon^2 v_{\tau\tau} &= \epsilon\left(v_t+\epsilon v_\tau-\frac{1}{3}(v_t+\epsilon v_\tau)^3\right), && (2.2)\\
&\quad\ 0<x<\pi,\ t>0,\\
v(0,t,\tau;\epsilon) &= 0,\ t\geq 0, && (2.3)\\
v_x(\pi,t,\tau;\epsilon) &= -\epsilon\left(m(v_{tt}(\pi,t)+2\epsilon v_{t\tau}(\pi,t)+\epsilon^2 v_{\tau\tau}(\pi,t))\right. && (2.4)\\
&\quad \left.+\gamma v(\pi,t)+\alpha(v_t(\pi,t)+\epsilon v_\tau(\pi,t))\right),\ t\geq 0,\\
v(x,0,0;\epsilon) &= \phi(x),\ 0<x<\pi, && (2.5)\\
v_t(x,0,0;\epsilon)+\epsilon v_\tau(x,0,0;\epsilon) &= \psi(x),\ 0<x<\pi. && (2.6)
\end{aligned}
$$

By expanding v into a power series with respect to ϵ around $\epsilon=0$, that is,

$$v(x,t,\tau;\epsilon)=v_o(x,t,\tau)+\epsilon v_1(x,t,\tau)+\epsilon^2 ...+\cdots, \tag{2.7}$$

and by substituting (2.7) into (2.2)-(2.6), and by equating the coefficients of like powers in ϵ, it follows from the power 0 and 1 of ϵ respectively, that v_0 should satisfy

$$
\begin{aligned}
v_{o_{tt}} - v_{o_{xx}} + p^2 v_o &= 0,\ 0 < x < \pi,\ t > 0, && (2.8)\\
v_o(0,t,\tau) &= 0,\ t \geq 0, && (2.9)\\
v_{o_x}(\pi,t,\tau) &= 0,\ t \geq 0, && (2.10)\\
v_o(x,0,0) &= \phi(x),\ 0 < x < \pi, && (2.11)\\
v_{o_t}(x,0,0) &= \psi(x),\ 0 < x < \pi, && (2.12)
\end{aligned}
$$

and that v_1 should satisfy

$$
\begin{aligned}
v_{1_{tt}} - v_{1_{xx}} + p^2 v_1 &= v_{o_t} - 2v_{o_{t\tau}} - \frac{1}{3}v_{o_t}^3,\ 0 < x < \pi,\ t > 0, && (2.13)\\
v_1(0,t,\tau) &= 0,\ t \geq 0, && (2.14)\\
v_{1_x}(\pi,t,\tau) &= -\left(m v_{o_{tt}}(\pi,t) + \gamma v_o(\pi,t) + \alpha v_{o_t}(\pi,t)\right),\ t \geq 0, && (2.15)\\
v_1(x,0,0) &= 0,\ 0 < x < \pi, && (2.16)\\
v_{1_t}(x,0,0) &= -v_{o_\tau}(x,0,0),\ 0 < x < \pi. && (2.17)
\end{aligned}
$$

The solution of (2.8) - (2.12) is given by

$$
v_o(x,t,\tau) = \sum_{n=0}^{\infty} \left(A_n \cos(\sqrt{\lambda_n} t) + B_n \sin(\sqrt{\lambda_n} t) \right) \sin((\tfrac{1}{2}+n)x), \qquad (2.18)
$$

where $\lambda_n = \left(\frac{1}{2}+n\right)^2 + p^2$, and where A_n and B_n are still arbitrary functions of τ which can be used to avoid secular terms in v_1. From (2.11), (2.12), and (2.18) it follows that $A_n(0)$ and $B_n(0)$ have to satisfy

$$
\begin{aligned}
A_n(0) &= \frac{2}{\pi}\int_0^{\pi} \phi(x)\sin((\tfrac{1}{2}+n)x)\,dx, && (2.19)\\
B_n(0) &= \frac{2}{\pi}\int_0^{\pi} \psi(x)\sin((\tfrac{1}{2}+n)x)\,dx, && (2.20)
\end{aligned}
$$

for $n = 0,\ 1,\ 2, \cdots$.

Next, we solve the initial-boundary value problem (2.13)-(2.17). In order to solve this problem we will make the boundary condition (2.15) homogeneous. For that reason we define the following transformation

$$
v_1(x,t,\tau) = w(x,t,\tau) - x(m v_{o_{tt}}(\pi,t,\tau) + \gamma v_o(\pi,t,\tau) + \alpha v_{o_t}(\pi,t,\tau)). \qquad (2.21)
$$

Substituting (2.21) into the initial-boundary value problem (2.13)-(2.17) we obtain

$$
\begin{aligned}
w_{tt} - w_{xx} + p^2 w &= v_{o_t} - 2v_{o_{t\tau}} - \frac{1}{3}v_{o_t}^3 && (2.22)\\
&\quad + x\left(f_{tt}(t,\tau) + p^2 f(t,\tau)\right),\ 0 < x < \pi,\ t > 0,
\end{aligned}
$$

$$
\begin{aligned}
w(0,t,\tau) &= 0,\ t \geq 0, && (2.23)\\
w_x(\pi,t,\tau) &= 0,\ t \geq 0, && (2.24)\\
w(x,0,0) &= xf(0,0), && (2.25)\\
w_t(x,0,0) &= -v_{o_\tau}(0,0) + xf_t(0,0), && (2.26)
\end{aligned}
$$

where $f(t,\tau) = mv_{o_{tt}}(\pi,t,\tau) + \gamma v_o(\pi,t,\tau) + \alpha v_{o_t}(\pi,t,\tau)$. It should be observed that $f_{tt}(t,\tau) + p^2 f(t,\tau) = \sum_{n=0}^{\infty}(-1)^n(\frac{1}{2}+n)^2\left(A_n^*\cos(\sqrt{\lambda_n}t) + B_n^*\sin(\sqrt{\lambda_n}t)\right)$, where $A_n^* = m\lambda_n A_n - \gamma A_n - \alpha\sqrt{\lambda_n}B_n$, and $B_n^* = m\lambda_n B_n - \gamma B_n + \alpha\sqrt{\lambda_n}A_n$.

To solve the initial-boundary value problem (2.22) - (2.26) the eigenfunction expansion method will be applied. For that reason, the function w is expanded into the Fourier series

$$
w(x,t,\tau) = \sum_{n=0}^{\infty} w_n(t,\tau)\sin((\tfrac{1}{2}+n)x). \tag{2.27}
$$

The function as defined in (2.27) satisfies the boundary conditions at $x = 0$ and $x = \pi$. By substituting (2.27) into (2.22) the left-hand side of (2.22) becomes,

$$
w_{tt} - w_{xx} + p^2 w = \sum_{n=0}^{\infty}(w_{n_{tt}} + \lambda_n w_n)\sin((\tfrac{1}{2}+n)x). \tag{2.28}
$$

By multiplying (2.22) with $\sin\left((\frac{1}{2}+n)x\right)$, and by integrating the so-obtained equation with respect to x from 0 to π, it follows that $w_n(t,\tau)$ has to satisfy

$$
\begin{aligned}
w_{n_{tt}} + \lambda_n w_n &= \frac{2(-1)^n}{\pi(n+\frac{1}{2})^2}\sum_{k=0}^{\infty}(-1)^k(\tfrac{1}{2}+n)^2\left(A_k^*\cos(\sqrt{\lambda_k}t) + B_k^*\sin(\sqrt{\lambda_k}t)\right)\\
&\quad +\sqrt{\lambda_n}\left([2A_n' - A_n]\sin(\sqrt{\lambda_n}t) + [B_n - 2B_n']\cos(\sqrt{\lambda_n}t)\right) \qquad (2.29)\\
&\quad -\frac{1}{4}\left(\sum_{\substack{k,l,m=0\\k+l-m=n}}^{\infty} - \sum_{\substack{k,l,m=0\\k-l-m-1=n}}^{\infty} - \frac{1}{3}\sum_{\substack{k,l,m=0\\k+l+m+1=n}}^{\infty}\right)H_k H_l H_m,
\end{aligned}
$$

where $H_n = \sqrt{\lambda_n}\left(-A_n\sin(\sqrt{\lambda_n}t) + B_n\cos(\sqrt{\lambda_n}t)\right)$. The last terms in the right-hand side of (2.29) (that is, the terms involving the sums) contain products of trigonometric functions. These products can be equal to $\sin(\sqrt{\lambda}t)$ or $\cos(\sqrt{\lambda}t)$, which are solutions of the homogeneous equation $w_{n_{tt}} + \lambda_n w_n = 0$. Obviously these products can give rise to secular terms in w, and so in v_1. To determine the terms in the products of the trigonometric functions that give rise to secular terms we have to solve the following Diophantine-like problems:

$$
k \quad +l - m = n, \text{or}\, k - l - m - 1 = n, \text{or}\, k + l + m + 1 = n, \tag{2.30}
$$

where

$$
\pm \quad \sqrt{\lambda_n} = \sqrt{\lambda_k} + \sqrt{\lambda_l} - \sqrt{\lambda_m}, \text{or} \tag{2.31}
$$

$$\pm \quad \sqrt{\lambda_n} = \sqrt{\lambda_k} - \sqrt{\lambda_l} + \sqrt{\lambda_m}, \text{or} \tag{2.32}$$
$$\pm \quad \sqrt{\lambda_n} = \sqrt{\lambda_k} - \sqrt{\lambda_l} - \sqrt{\lambda_m}, \text{or} \tag{2.33}$$
$$\sqrt{\lambda_n} = \sqrt{\lambda_k} + \sqrt{\lambda_l} + \sqrt{\lambda_m}, \tag{2.34}$$

with k, m, l, and n in $\mathbb{N}$, and $p^2 > 0$. Note that $\lambda_j = (\frac{1}{2}+j)^2 + p^2$. To solve these problems (2.30) - (2.34) we use a technique similar to the one used in [12].

By substituting (2.30) (that is, $k+l-m = n,$, or $k-l-m-1 = n$, or $k+l+m+1 = n$) into (2.31), or (2.32), or (2.33), or (2.34), by squaring the so-obtained equation twice, by rearranging terms and by using some elementary algebraic manipulations we find that the Diophantine-like problems (2.30) - (2.34) only have solutions for

1. $n = k+l-m$ and $\sqrt{\lambda_n} = \sqrt{\lambda_k} + \sqrt{\lambda_l} - \sqrt{\lambda_m}$. In this case the solution is given by: $l = m$ and $n = k$, or $k = m$ and $n = l$.

2. $n = k+l-m$ and $\sqrt{\lambda_n} = \sqrt{\lambda_k} - \sqrt{\lambda_l} + \sqrt{\lambda_m}$. In this case the solution of the equation is given by $l = m$ and $n = k$.

3. $n = k+l-m$ and $\sqrt{\lambda_n} = -\sqrt{\lambda_k} + \sqrt{\lambda_l} + \sqrt{\lambda_m}$. In this case the solution of the equation is given by $k = m$ and $n = l$.

We rewrite (2.29) by taking apart those terms in the right-hand side of (2.29) that give rise to secular terms in w, yielding

$$\begin{aligned}
w_{n_{tt}} + \quad \lambda_n \quad w_n = & \Bigg(2\sqrt{\lambda_n} A_n' - \sqrt{\lambda_n} A_n - \frac{2}{\pi}(-m\lambda_n B_n + \gamma B_n - \alpha\sqrt{\lambda_n} A_n) \\
& - \frac{1}{4}\left(\frac{1}{4}\lambda_n \sqrt{\lambda_n} A_n (A_n^2 + B_n^2) - \sqrt{\lambda_n} A_n \sum_{m=0}^{\infty} \lambda_m (A_m^2 + B_m^2) \right) \Bigg) \sin(\sqrt{\lambda_n} t) \\
& + \Bigg(-2\sqrt{\lambda_n} B_n' + \sqrt{\lambda_n} B_n - \frac{2}{\pi}(-m\lambda_n A_n + \gamma A_n + \alpha\sqrt{\lambda_n} B_n) \\
& - \frac{1}{4}\left(-\frac{1}{4}\lambda_n \sqrt{\lambda_n} B_n (A_n^2 + B_n^2) + \sqrt{\lambda_n} B_n \sum_{m=0}^{\infty} \lambda_m (A_m^2 + B_m^2) \right) \Bigg) \cos(\sqrt{\lambda_n} t) \\
& - \frac{1}{4}\left(\sideset{}{^*}\sum_{\substack{k,l,m=0 \\ k+l-m=n}}^{\infty} - \sum_{\substack{k,l,m=0 \\ k-l-m-1=n}}^{\infty} - \frac{1}{3} \sum_{\substack{k,l,m=0 \\ k+l+m+1=n}}^{\infty} \right) H_k H_l H_m \\
& + \frac{2(-1)^n}{\pi(n+\frac{1}{2})^2} \sum_{\substack{k=0 \\ k\neq n}}^{\infty} (-1)^k (k+\tfrac{1}{2})^2 \left(A_k^* \cos(\sqrt{\lambda_k} t) + B_k^* \sin(\sqrt{\lambda_k} t) \right),
\end{aligned} \tag{2.35}$$

where the * in $\sum^*$ indicates that terms in this sum giving rise to secular terms are excluded. In order to avoid secular terms in w (and in v_1) we have to take the coefficients of $\sin(\sqrt{\lambda_n} t)$

and $\cos(\sqrt{\lambda_n}t)$ in the right-hand side of (2.35) to be equal to zero, yielding

$$\begin{aligned} 2\sqrt{\lambda_n}A_n' &- \sqrt{\lambda_n}A_n = \frac{2}{\pi}(-m\lambda_n B_n + \gamma B_n - \alpha\sqrt{\lambda_n}A_n(\tau)) \\ &+\frac{1}{4}\left(\frac{1}{4}\lambda_n\sqrt{\lambda_n}A_n(A_n^2+B_n^2) - \sqrt{\lambda_n}A_n\sum_{m=0}^{\infty}\lambda_m(A_m^2+B_m^2)\right), \quad (2.36) \\ 2\sqrt{\lambda_n}B_n' &- \sqrt{\lambda_n}B_n = -\frac{2}{\pi}(-m\lambda_n A_n + \gamma A_n + \alpha\sqrt{\lambda_n}B_n(\tau)) \\ &+\frac{1}{4}\left(\frac{1}{4}\lambda_n\sqrt{\lambda_n}A_n(A_n^2+B_n^2) - \sqrt{\lambda_n}A_n\sum_{m=0}^{\infty}\lambda_m(A_m^2+B_m^2)\right) \quad (2.37) \end{aligned}$$

for $n = 0, 1, 2, \cdots$. By taking $\sqrt{\lambda_n}A_n = \bar{A}_n$, $\sqrt{\lambda_n}B_n = \bar{B}_n$ system (2.36) - (2.37) simplifies to

$$\begin{aligned} 2\bar{A}_n' - \bar{A}_n &= \frac{2}{\pi}\left((-m\sqrt{\lambda_n} + \frac{\gamma}{\sqrt{\lambda_n}})\bar{B}_n - \alpha\bar{A}_n\right) \quad (2.38) \\ &+\frac{1}{4}\left(\frac{1}{4}\bar{A}_n(\bar{A}_n^2+\bar{B}_n^2) - \bar{A}_n\sum_{m=0}^{\infty}(\bar{A}_m^2+\bar{B}_m^2)\right), \\ 2\bar{B}_n' - \bar{B}_n &= -\frac{2}{\pi}\left((-m\sqrt{\lambda_n} + \frac{\gamma}{\sqrt{\lambda_n}})\bar{A}_n + \alpha\bar{B}_n\right) \quad (2.39) \\ &+\frac{1}{4}\left(\frac{1}{4}\bar{B}_n(\bar{A}_n^2+\bar{B}_n^2) - \bar{B}_n\sum_{m=0}^{\infty}(\bar{A}_m^2+\bar{B}_m^2)\right) \end{aligned}$$

for $n = 0, 1, 2, \cdots$. From (2.38) and (2.39) it can easily be seen that if $A_n(0) = B_n(0) = 0$ then $A_n(\tau) = B_n(\tau) = 0$ for $\tau = \epsilon t > 0$. So if we start with no initial energy in the n-th mode then there will be no energy present up to $O(\epsilon)$ on timescales of order ϵ^{-1}. This allows us to truncate the infinite dimensional system (2.38) - (2.39) to those modes which have non-zero initial energy. To study system (2.38) - (2.39) in more detail we will use polar coordinates as defined by

$$\begin{aligned} \bar{A}_n &= R_n\cos(\phi_n), \quad (2.40) \\ \bar{B}_n &= R_n\sin(\phi_n), \quad (2.41) \end{aligned}$$

where R_n and ϕ_n are functions of τ.

After substituting (2.40) - (2.41) into (2.38) - (2.39) we obtain

$$\begin{aligned} R_n' &= \frac{R_n}{2}\left(1 - \frac{2}{\pi}\alpha + \frac{1}{16}R_n^2 - \frac{1}{4}\sum_{m=0}^{\infty}R_m^2\right), \quad (2.42) \\ \phi_n' &= \frac{1}{\pi}\left(m\sqrt{\lambda_n} - \frac{\gamma}{\sqrt{\lambda_n}}\right) \quad (2.43) \end{aligned}$$

for $n = 0, 1, 2, \cdots$. From (2.42) it is obvious that $R'_n < 0$ for $\alpha > \frac{\pi}{2}$. So for $\alpha > \frac{\pi}{2}$ all solutions of (1.2) - (1.6) will tend to zero for increasing time t. When for instance only energy is initially present in the first two modes (that is, $R_o(0) \neq 0$, $R_1(0) \neq 0$, and $R_n(0) = 0$ for $n \geq 2$) a phase-plane analysis can be performed.

From (2.42) it then follows that R_o and R_1 have to satisfy

$$R'_o = \frac{R_o}{2}\left(1 - \frac{2}{\pi}\alpha - \frac{3}{16}R_o^2 - \frac{1}{4}R_1^2\right), \quad (2.44)$$

$$R'_1 = \frac{R_1}{2}\left(1 - \frac{2}{\pi}\alpha - \frac{1}{4}R_o^2 - \frac{3}{16}R_1^2\right). \quad (2.45)$$

The critical points of (2.44) and (2.45) for $0 < \alpha < \frac{\pi}{2}$ are $(0,0)$, $\left(\frac{4}{3}\sqrt{\frac{3}{\pi}(\pi - 2\alpha)}, 0\right)$, $\left(0, \frac{4}{3}\sqrt{\frac{3}{\pi}(\pi - 2\alpha)}\right)$, and $\left(\frac{4}{7}\sqrt{\frac{7}{\pi}(\pi - 2\alpha)}, \frac{4}{7}\sqrt{\frac{7}{\pi}(\pi - 2\alpha)}\right)$. For $\alpha \geq \frac{\pi}{2}$ the only critical point is $(0,0)$. By linearizing (2.44) and (2.45) around the critical points for $0 < \alpha < \frac{\pi}{2}$ we find two stable nodes, one unstable node, and one saddle point. For $\alpha > \frac{\pi}{2}$ the critical point is a stable node (see table 1).

Table 1: The behaviour of the critical points

α	Critical point	Behaviour
$0 < \alpha < \frac{\pi}{2}$	(0,0)	unstable node
	$\left(0, \frac{4}{3}\sqrt{\frac{3}{\pi}(\pi - 2\alpha)}\right)$	stable node
	$\left(\frac{4}{3}\sqrt{\frac{3}{\pi}(\pi - 2\alpha)}, 0\right)$	stable node
	$\left(\frac{4}{7}\sqrt{\frac{7}{\pi}(\pi - 2\alpha)}, \frac{4}{7}\sqrt{\frac{7}{\pi}(\pi - 2\alpha)}\right)$	saddle point
$\alpha > \frac{\pi}{2}$	$(0,0)$	stable node

From the table it can readily be seen that if the damping parameter α is increasing (starting from $\alpha = 0$) then the two stable nodes and the saddle point are moving to the unstable node. For $\alpha = \frac{\pi}{2}$ the four critical points coincide in $(0,0)$, and for $\alpha > \frac{\pi}{2}$ a stable node occurs in $(0,0)$. The behaviour of the solution of (2.44) - (2.45) can also be seen in figure 2 and in figure 3.

For $0 < \alpha < \frac{\pi}{2}$ it can be seen in figure 2 that the solution (usually) will finally tend to a single mode vibration as $t \to \infty$. For $\alpha > \frac{\pi}{2}$ it can be seen in figure 3 that the string vibrations will finally come to rest up to $O(\epsilon)$ as $t \to \infty$.

After removing secular terms in (2.35) we can finally determine w_n from (2.35). After

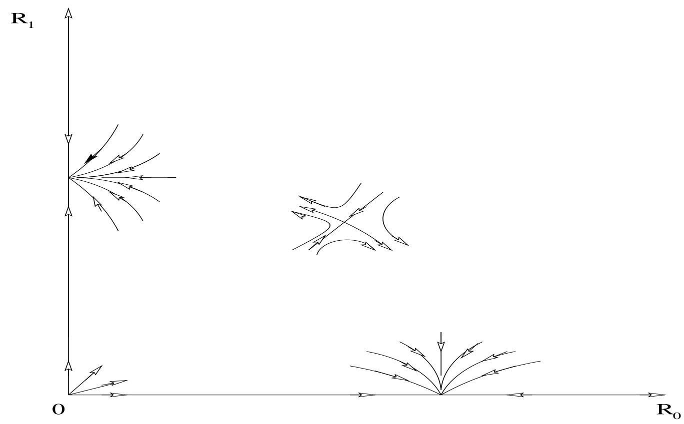

Figure 2: Phase Plane for $0 < \alpha < \frac{\pi}{2}$.

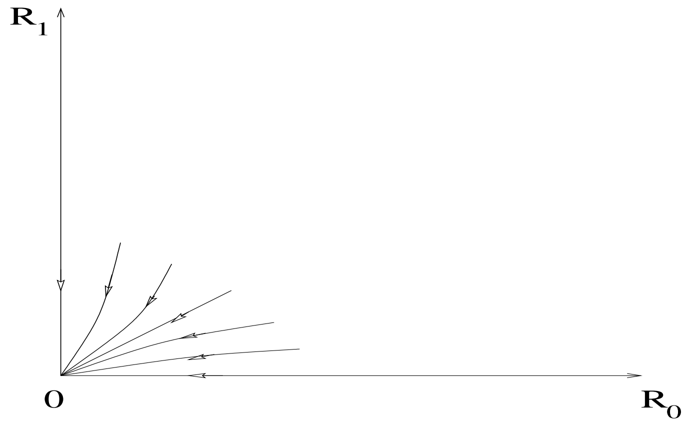

Figure 3: Phase plane for $\alpha > \frac{\pi}{2}$.

some lengthy, but elementary calculations we obtain

$$w_n \quad (t,\tau) = F_n(t,\tau) + \sum_{\substack{k=0\\k\neq n}}^{\infty} g_{nk}\left(A_k^* \cos(\sqrt{\lambda_k}t) + B_k^* \sin(\sqrt{\lambda_k}t)\right) \qquad (2.46)$$

$$- \quad \frac{1}{4}\left(\sum_{\substack{k,l,m=0\\k+l-m=n}}^{\infty *} - \sum_{\substack{k,l,m=0\\k-l-m-1=n}}^{\infty} - \frac{1}{3}\sum_{\substack{k,l,m=0\\k+l+m+1=n}}^{\infty}\right) S_{klm} \sum_{i=1}^{4} \frac{\cos(T_{klm}^i t + \delta_{klm}^i)}{\lambda_n - (T_{klm}^i)^2},$$

where $F_n(t,\tau) = C_n(\tau)\cos(\sqrt{\lambda_n}t) + D_n(\tau)\sin(\sqrt{\lambda_n}t)$,

$$A_n^* = m\lambda_n A_n - \gamma A_n - \alpha\sqrt{\lambda_n}B_n,$$

$$B_n^* = m\lambda_n B_n - \gamma A_n + \alpha\sqrt{\lambda_n}A_n,$$

$$g_{nk} = \frac{2}{\pi}\left(\frac{k+\frac{1}{2}}{n+\frac{1}{2}}\right)^2 \frac{(-1)^{k+n}}{\lambda_n-\lambda_k},$$

$$S_{klm} = \sqrt{\lambda_k\lambda_l\lambda_m}\prod_{i=k,l,m}\sqrt{A_i^2(\tau)+B_i^2(\tau)},$$

$$T^1_{klm} = \sqrt{\lambda_k}+\sqrt{\lambda_l}+\sqrt{\lambda_m}, \qquad \delta^1_{klm} = \alpha_k+\alpha_l+\alpha_m,$$

$$T^2_{klm} = \sqrt{\lambda_k}+\sqrt{\lambda_l}-\sqrt{\lambda_m}, \qquad \delta^2_{klm} = \alpha_k+\alpha_l-\alpha_m,$$

$$T^3_{klm} = T^2_{kml}, \qquad \delta^3_{klm} = \delta^2_{kml},$$

$$T^4_{klm} = \sqrt{\lambda_k}-\sqrt{\lambda_l}-\sqrt{\lambda_m}, \qquad \delta^4_{klm} = \alpha_k-\alpha_l-\alpha_m,$$

and where α_n is defined as follows:

for $A_n^2+B_n^2=0 : \alpha_n = 0$,
for $A_n^2+B_n^2 \neq 0 : \cos(\alpha_n) = \frac{B_n}{\sqrt{A_n^2+B_n^2}}$, and $\sin(\alpha_n) = \frac{A_n}{\sqrt{A_n^2+B_n^2}}$.

It should be observed that w_n still contains infinitely many free functions C_n and D_n of τ for $n = 0, 1, 2, \cdots$. These functions can be used to avoid secular terms in the solution of the $O(\epsilon^2)$-problem for v_2. It is, however, our goal to construct a function $\bar{u}$ that satisfies the partial differential equation, the boundary conditions, and the initial values up to order ϵ^2. For that reason C_n and D_n are taken to be equal to their initial values $C_n(0)$ and $D_n(0)$ respectively. So far we constructed a formal approximation $\bar{u} = v_o + \epsilon v_1$ for u that satisfies the partial differential equation, the boundary conditions, and the initial values up to order ϵ^2. In [4,10,12,13] asymptotic theories are presented for wave and beam equations with similar nonlinearities. The formal approximations constructed for those problems were shown to be asymptotically valid, i.e., the differences between the approximations and the exact solutions are of order ϵ on timescales of order ϵ^{-1} as $\epsilon \to 0$. It is beyond the scope of this paper to give the asymptotic analysis for the wave equation we discussed. We expect that the asymptotic validity of the constructed approximations can be shown in a way similar to the analysis presented in [4,10,12,13]. Finally it should be remarked that from these asymptotic theories it follows that $v_o+\epsilon v_1$ and v_o are both (order ϵ) asymptotic approximations of the exact solution on timescales of order ϵ^{-1}.

3 Conclusions

In this paper an initial-boundary value problem for a weakly nonlinear wave equation with a non-classical boundary condition has been considered. Formal asymptotic approximations of the exact solution have been constructed using a two-timescales perturbation method. For all values of $p^2 > 0$ it has been shown that mode-interactions only occur between modes with non-zero initial energy up to $O(\epsilon)$. This implies that truncation is allowed to those modes that have non-zero initial energy up to $O(\epsilon)$. For the damping parameter

$\alpha > \frac{\pi}{2}$ it also has been shown that all solutions tend to zero as $t \to \infty$. For $0 < \alpha < \frac{\pi}{2}$ it can be shown that the string system usually will oscillate in only one mode (up to $O(\epsilon)$) as $t \to \infty$.

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