Delta Method for Deriving the Consistency of Bootstrap Estimator

 for Parameter of Autoregressive Model

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**Abstract.** Let  be the first order of autoregressive model and let  be the sample that satisfies such model, i.e. the sample follows the relation  where  is a zero mean white noise process with constant variance . Let  be the estimator for parameter . Brockwell and Davis (1991) showed that  and . Meantime, Central Limit Theorem asserts that the distribution of  converges to Normal distribution with mean *0* and variance  as . In bootstrap view, the key of bootstrap terminology says that the population is to the sample as the sample is to the bootstrap samples. Therefore, when we want to investigate the consistency of the bootstrap estimator for sample mean, we investigate the distribution of  contrast to , where is a bootstrap version of  computed from sample bootstrap . Asymptotic theory of the bootstrap sample mean is useful to study the consistency for many other statistics. Let  be the bootstrap estimator for . In this paper we study the consistency of  using delta Method. After all, we construct a measurable map  such that  =  conditionally almost surely, by applying the fact that, where *G* is a normal distribution. We also present the Monte Carlo simulations to emphisize the conclusions.

***Keywords****:* Bootstrap, consistency, autoregressive model, delta method, Monte Carlo simulations

1. **Introduction**

Studying of estimation of the unknown parameter  involves: (1) what estimator  should be used? (2) having choosen to use particular , is this estimator consistent to the population parameter ? (3) how accurate is  as an estimator for true parameter ? (4) the interesting one is, what is the asymptotic distribution of such estimator? The bootstrap is a general methodology for answering the second and third questions, while the delta method is used to answer the last question. Consistency theory is needed to ensure that the estimator is consistent to the actual parameter as desired.

Consider the parameter  is the population mean. The consistent estimator for  is the sample mean . The consistency theory is then extended to the consistency of bootstrap estimator for mean. According to the bootstrap terminology, if we want to investigate the consistency of bootstrap estimator for mean, we investigate the distribution of  and . The consistency of bootstrap under Kolmogorov metric is defined as

 (1)

Bickel and Freedman (1981) and Singh (1981) showed that (1) converges almost surely to zero as. The consistecy of bootstrap for mean is a worthy tool for studying the consistency and limiting distribution of other statistics. In this paper, we study the asymptotic distribution of , i.e. bootstrap estimator for parameter of the AR(1) process. Suprihatin, *et.al* (2013) also studied the advantage of bootstrap for estimating the median, and the results gave a good accuracy.

The consistency of bootstrap estimator for mean is then applied to study the asymptotic distribution of , i.e. bootstrap estmate for parameter of the AR(1) process using delta method. We describe the consistency of bootstrap estimates for mean and investigate the limiting distribution of . Section 2 reviews the consistency of bootstrap estimate for mean under Kolmogorov metric and describe the estimation of autocovariance function. Section 3 deal with asymptotic distribution of  using delta method. Section 4 discuss the results of Monte Carlo simulations involve bootstrap standard errors and density estmation for mean and . Section 5, is the last section, briefly describes the conclusions of the paper.

1. **Consistency of Bootstrap Estimator For Mean and Estimation of Autocovariance Function**

Let  be a random sample of size *n* from a population with common distribution *F* and let  be the specified random variable or statistic of interest, possibly depending upon the unknown distribution *F.* Let  denote the empirical distribution function of , i.e., the distribution putting probability 1/*n* at each of the points . The bootstrap method is to approximate the distribution of  under *F* by that of  under  whrere  denotes a bootstrapping random sample of size *n* from .

We start with definition of consistency. Let *F* and *G*  be two distribution functions on sample space ***X***. Let  be a metric on the space of distribution on ***X***. For  i.i.d from *F,* and a given functional , let

,

.

We say that the bootstrap is consistent (strongly) under  for *T* if 

 Let functional *T* is defined as  where  and  are sample mean and population mean respectively. Bootstrap version of *T*  is , where  is boostrapping sample mean. Bootstrap method is a device for estimating  by . Singh (198) studied the consistency of bootstrap under Kolmogorov metric,

 = 

Meanwhile, Bickel and Freedman (1981) studied the same topic but tey used Mallows metric. The crux result of both papers is that . Suprihatin, *et.al* (2011) emphasized this result by giving nice simulations and agree with their results. Papers of Singh (198) and Bickel and Freedman (1981) have become the foundation for studying other complicated statistics.

 Suppose we have the observed values  from the stationary AR(1) process. A natural estimators for parameters mean, covariance and correlation function are , , and  respectively. If the series  is replaced by the centered series , then the autocovariance function does not change. Therefore, studying the asymptotic properties of the sample autocovariance function , it is not a loss of generality to assume that  = 0. The sample autocovariance function can be written as

. (2)

Under some conditions (see, e.g., van der Vaart (2012)), the last three terms in (2) is of the order . Thus, under assumption that  = 0, we can write (2) in simple notation,

.

The asymptotic behaviour of the sequence  depends only on . Note that a change of  by  is asymptotically negligible, so that, for simplicity of notation, we can equivalently study the average

.

Both  and  are unbiased estimator of , under the condition that . Their asymptotic distribution then can be derived by applying a central limit theorem to the average  of the variables . The asymptotic variance takes the form  and in general depends on fourth order moments of the type  as well as on the autocovariance function of the series . Van der Vaart (2012) showed that the autocovariance function of the series  can be written as

 (3)

Where  the fourth cumulant of . The following theorem give the asymptotic distribution of the sequence .

**Theorem 1** *If* *holds for an i.i.d. sequence*  *with mean zero and*  *and numbers*  *with*  *then* .

1. **Asymptotic Distribution of Bootstrap Estimate For Parameter of AR(1) Process Using Delta Method**

The delta method consists of using a Taylor expansion to approximate a random vector of the form  by the polynomial  in . This method is useful to deduce the limit law of  from that of , which is guaranteed by the next theorem.

**Theorem 2** *Let*   *be a map defined on a subset of*  *and differentiable at* . *Let*  *be random vectors taking their values in the domain of* . *If*  *for numbers* , *then* . *Moreover, the difference between*  *and*  *converges to zero in probability.*

Assume that  is a statistic, and that  is a given differensiable map. The bootstrap estimator for the distribution of  is . If the bootstrap is consistent for estimating the distribution of , then it is also consistent for estimating the distribution of , as given in the following theorem. The proof of the theorem is due to van der Vaart (2000).

**Theorem 3** (Delta Method For Bootstrap)*Let*  *be a measurable map defined and continuously differentiable in a neighborhood of* . *Let*  *be random vectors taking their values in the domain of*  *that converge almost surely to* . *If*  *and*  *conditionally almost surely, then both*  *and*  *conditionally almost surely.*

**Proof**. By applying the mean value theorem, the difference  can be written as  for a point  between  and , if the latter two points are in the ball around  in which  is continuously differentiable. By the continuity of the derivative, there exists a constant  for every  such that  <  for every *h* and every . If *n* is suffeciently large,  suffeciently small, , and , then



 .

Fix a number  and a large number *M.* For  sufficiently small to ensure that ,

. (4)

Since , the right side of (4) converges almost surely to  for every continuity point *M* of . This can be made arbitrarily small by choice of *M.* Conclude that the left side of (4) converges to zero almost surely. The theorem follows by an application of Slutsky’s lemma. ■

For the AR(1) process, from Yule-Walker equation we obtain the moment estimator  where  be the lag 1 sample autocorrelation

. (5)

According to Davison and Hinkley (2006), the estimate of standard error of parameter  is $\hat{se}\left(θ\right)=$ . Meanwhile, the bootstrap version of standard error was introduced by Efron and Tibshirani (1986). In Section 4 we demonstrate results of Monte Carlo simulations consist the two of standard errors and give brief comments. In Suprihatin, *et.al.* (2012) we construct a measurable function  as follows. Equation (5) can be written as

 







Brockwell and Davis (1991) have shown that  is consistent estimator of true parameter . Kolmogorov SLLN asserts that . Since  is independent of , then  = 0. Hence, . By applying the Slutsky’s lemma, the last display is approximated by  Thus, for  we obtain . We see that  equals to  for the function. Since  is continous and hence is measurable. Suppose that is based on a sample from a distribution with finite first four moments of  By central limit theorem and applying Theorem 1 we conclude that

,

where  as in (3) for . The map is differentiable at the point , with derivative . Theorem 2 says that

 =  + 

 =  + .

In view of Theorem 2, if *T* possesses the normal with mean 0 and variance , then

 ~  .

Meantime, the bootstrap version of , denoted by  can be obtained as follows [see, *e.g*. Efron dan Tibshirani (1986) and Freedman (1985)]:

1. Define the residuals  for 
2. A bootstrap sample  is created by sampling  with replacement from the residuals. Letting  as an initial bootstrap sample and , .
3. Finally, after centering the bootstrap time series  i.e.  is replaced by  where . Using the *plug-in* principle, we obtain the bootstrap estimator  computed from the sample  .

Analog with the previous discussion, we obtain the bootstrap version for counterpart of , that is measurable map  Thus, in view of Theorem 3 we conclude that  converges to  conditionally almost surely. By the Glivenko-Cantelli lemma and applying the *plug-in* principle, we obtain



and

.

1. **Results of Monte Carlo Simulations**

The simulation is conducted using S-Pus and the sample is the 50 time series data of exchange rate of US dollar compared to Indonesian rupiah. Data is taken from authorized website of Bank Indonesia, i.e. <http://www.bi.go.id> for fifty days of transactions on March and April 2012. Suprihatin, *et. al*. (2012) has identified that the time series data satisfies the AR(1) procces, such that the data follows the equation



where $\~$ WN. The simulation yields the estimator  = - 0.448 with standard error 0.1999. To produce a good approximation, Efron and Tibshirani (1986) and Davison and Hinkley (2006) suggest to use the number of resamples at least *B =* 50. Bootstrap version of standard errror using bootstrap samples of size *B =* 25, 50, 100, 1,000 and 2,000 yielding as presented in Table 1.

**Table 1** Estimates for Standard Errors of  for Various *B*

|  |  |
| --- | --- |
|  | *B* |
| 25 | 50 | 100 | 500 | 1,000 | 2,000 |
|  | 0.2005 | 0.1981 | 0.1997 | 0.1991 | 0.1972 | 1.1964 |

 From Table 1 we can see that the values of bootstrap standard errors tend to decrease in term of size of *B* increase and closed to the value of 0.1999 (actual standard error). These results show thatthe bootstrap gives a good estimate. Meantime, the histogram and density estimate of  are presented in Figure 1. From Figure 1 we can see that the resulting histogram close related to the normal density. Of course, this result agree to the result of Freedman (1985) and Bose (1988).



**Figure 1** Histogram and Density Estimate of Bootstrap Estimator 

1. **Conclusions**

A number of points arise from the study of Section 2, 3, and 4, amongst which we state as follows.

1. Consider an AR(1) process  with Yule-Walker estimator  =  is a consistent estimator for the true parameter . By using the delta method we have shown that  is also a consistent estimator for  and

  for . Moreover, we obtain the crux result that

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1. Resulting of Monte Carlo simulations show that the bootstrap estimators are good approximations, as represented by their standard errors and plot of densities estimation.

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