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Deriving Conformal Group from Infinitesimal Displacement Field in Minkowski Space

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Abstract : Conformal group, which conserves causal structure only, has been derived from the displacement field of infinitesimal transformations in the Minkowski space. We can derive this group using the transformation formula connecting two coordinate systems, which involves 15 parameters taking the roles as the group elements. In addition we consider the group representations of the corresponding linear transformations accommodated in a 6-dimensional real space.

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1 INTRODUCTION

It is well known that the conformal group plays a significant role in the quantum field theory of high energy physics, for instance, the conformal symmetry related to it is needed in hadronic physics [1]. Furthermore, there have been mathematical and physical investigations that imply the conformal group. For examples, the conformal group in multi dimensional space-time [2, 3, 4, 5], the various connections between the conformal group and the quantum theory or quantum gravity [6, 7, 8], and possible connection with perfect fluid [9, 10].

Emerging from advanced investigations either mathematically or physically as explained above, with a novelty level that would be presented here, there is a fundamental pedagogical need related to how the conformal group has been derived. In other words, how can the 15 parameters of the group be easily obtained and become elements of the group in a comprehensible but still representative way?

Generally, the conformal group corresponding to the conformal transformations which preserve certain internal characters, is the 15-parameter Lie group representing the set of operators that transform objects from a flat space to an other flat space [7]. This group consists of the space-time translations (4 parameters), the proper homogeneous Lorentz transformations (6 parameters), the dilatations (1 parameter), and the conformal transformations (4 parameters).

Taking care of the difficulties of an explaining pedagogically how to extract these 15 parameters, in this paper we will carry out an exploration about the conformal group that according to us could be presented in a pretty simple way. It will be derived from the displacement field of infinitesimal transformations in the Minkowski space [11]. The derivation will be done using a transformation formula between two coordinate systems. To establish this, we will consider a 4-dimensional space-time manifold which will be divided into three regions spanning the collection of points in the neighbourhood of a specific point x, *i.e.*, the forward cone V^+, the backward cone V^-, and the complement S of these two cones which accommodates the space-like regions†. All events in the first two regions have the causal relationship with x. The last region contains all points which can have no causal relationship with x. The boundary of V^+ is formed by all possible events which can be reached by a light signal sent from x. The problem involved, therefore, deals with the locality principle which is emphasized by Einstein causality principle that *no physical effect can propagate faster than light*.

In special relativity one assumes that the causal structure of space-time is an ápriori globally given attribute. There is supposed to exist a preferred

†This division is based on the fundamental analysis leading to special relativity; the comparison of times at different places is not an objectively well defined procedure if all laws are local and speed of light is limited.

class of coordinate systems, the "inertial systems", in which the causal future of a point $x = (t, x)$ consists of all points $x' = (t', x')$ satisfying [11],

$$t' - t \geq |\mathbf{x}' - \mathbf{x}| \ . \tag{1}$$

The Lorentz distance squared of two space-time points x^μ and x'^μ is given by

$$\Delta x^2 = g_{\mu\nu} \left(x'^\mu - x^\mu \right) \left(x'^\mu - x^\mu \right) \ , \tag{2}$$

where $g_{\mu\nu}$ is the metric tensor of Minkowski real space-time,

$$g_{\mu\nu} = (-1)^{\delta_{\overline{\mu}0}} \delta_{\overline{\mu}\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ , \tag{3}$$

where bars above doubly repeated subscripts negates summation. Eq.(2) has positive values for all time-like separations, negative for space-like separations, and null if x and x' can be connected by a light signal. The 4-dimensional continuum defined by Eq.(2) with the metric given by Eq.(3) is called Minkowski space and will be denoted by $\mathcal{M}$.

2 ELABORATION AND DISCUSSION

Conformal group is the group that conserves only the causal structure, but not the metric [11]. To obtain the group elements, we will apply a formula of the transformation between two coordinate systems in which the light cones are characterized by $\Delta x^2 = 0$

Let us consider a small neighborhood of a point x and an infinitesimal coordinate transformation given by,

$$x^\mu \longrightarrow x^\mu + \epsilon u^\mu(x), \qquad \epsilon \longrightarrow 0 \ . \tag{4}$$

The element of line segment from x to $x+dx$ having a "length" squared,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \ , \tag{5}$$

changes by,

$$\begin{aligned} \delta ds^2 &= g_{\mu\nu} \delta(dx^\mu dx^\nu) \\ &= g_{\mu\nu}((\underbrace{\delta dx^\mu}_{=\epsilon du^\mu}) dx^\nu + dx^\mu \underbrace{\delta dx^\nu}_{=\epsilon du^\nu}) \\ &= \epsilon g_{\mu\nu} \left((\partial_\rho u^\mu) dx^\rho dx^\nu + (\partial_\rho u^\nu) dx^\rho dx^\mu \right) \\ &= \epsilon G_{\mu\nu}(x) dx^\mu dx^\nu \ , \end{aligned} \tag{6}$$

where $\partial_\rho \equiv \partial / \partial x^\rho$ is the covariant derivative operator (the gradient) in a flat space and,

$$(a) \quad G_{\mu\nu}(x) = \partial_\nu u_\mu + \partial_\mu u_\nu ; \qquad (b) \quad u_\mu = g_{\mu\nu} u^\nu \ . \tag{7}$$

In Eqs.(5), (6), and (7b) $g_{\mu\nu}$ is a constant. If no change of the light-cone equation is caused by coordinate transformations, than one must have,

$$G_{\mu\nu}(x) dx^\mu dx^\nu = 0 \quad \text{whenever} \quad g_{\mu\nu} dx^\mu dx^\nu = 0 \ . \tag{8}$$

Thus $G_{\mu\nu}$ behaves as a metric tensor having like-light direction the same as $g_{\mu\nu}$, thereby $G_{\mu\nu}$ can be written as,

$$G_{\mu\nu}(x) = \lambda(x) g_{\mu\nu} \ . \tag{9}$$

Furthermore, $\lambda(x)$ can be eliminated by contracting Eq.(9) doubly with $g^{\mu\nu}$ producing,

$$\begin{aligned} G_{\mu\nu}(x) g^{\mu\nu} &= \lambda(x) g_{\mu\nu} g^{\mu\nu} \\ &= \lambda(x) \left(g_{00} g^{00} + g_{11} g^{11} + g_{22} g^{22} \right. \\ &\qquad \left. + g_{33} g^{33} \right) \\ &= 4\lambda(x) \ , \end{aligned} \tag{10}$$

or $\lambda(x) = (1/4) \, G_{\mu\nu} g^{\mu\nu}$. From Eqs.(7), (9), and (10) we can obtain,

$$\begin{aligned} G_{\mu\nu}(x) - \frac{1}{4} G_{\rho\sigma} g^{\rho\sigma} g_{\mu\nu} &= 0 \\ (\partial_\nu u_\mu + \partial_\mu u_\nu) - \frac{1}{4} \underbrace{(\partial_\sigma u_\rho + \partial_\rho u_\sigma) \, g^{\rho\sigma}}_{\partial_\sigma u^\sigma + \partial_\rho u^\rho = 2\partial_\rho u^\rho} g_{\mu\nu} &= 0 \\ (\partial_\nu u_\mu + \partial_\mu u_\nu) - \frac{1}{2} g_{\mu\nu} \partial_\rho u^\rho &= 0 \ , \end{aligned} \tag{11}$$

which is a differential equation of the displacement field u related to an infinitesimal transformation.

A general solution of Eq.(11) can be obtained by using Maclaurin series expansion method (a Taylor expansion around the point $x = 0$) in which the displacement field u is to be written in the form,

$$\begin{aligned} u_\mu(x) &= a^{(1)}_\mu + a^{(2)}_{\mu\lambda_1} x^{\lambda_1} + a^{(3)}_{\mu\lambda_1\lambda_2} x^{\lambda_1} x^{\lambda_2} + \cdots \\ &\quad + a^{(n)}_{\mu\lambda_1\cdots\lambda_{n-1}} x^{\lambda_1} \cdots x^{\lambda_{n-1}} + \cdots \ , \end{aligned} \tag{12}$$

with $a^{(n)}_{\mu\lambda_1\cdots\lambda_{n-1}}$ symmetrical in $(\lambda_1 \cdots \lambda_{n-1})$. By using Eq.(12) the first term in Eq.(11) can be expressed as,

$$\begin{aligned} \partial_\nu u_\mu &= a^{(2)}_{\mu\lambda_1} \partial_\nu x^{\lambda_1} \\ &\quad + a^{(3)}_{\mu\lambda_1\lambda_2} \left((\partial_\nu x^{\lambda_1}) x^{\lambda_2} + x^{\lambda_1} \partial_\nu x^{\lambda_2} \right) + \cdots \\ &\quad + a^{(n)}_{\mu\lambda_1\cdots\lambda_{n-1}} \left((\partial_\nu x^{\lambda_1}) x^{\lambda_2} \cdots x^{\lambda_{n-1}} + \cdots \right. \\ &\quad \left. + x^{\lambda_1} \cdots x^{\lambda_{n-2}} \partial_\nu x^{\lambda_{n-1}} + \cdots \right) \ . \end{aligned} \tag{13}$$

For each term in this equation, the covariant derivatives do not vanish only if the subscript of ∂ is the same as the superscript of x since

$\partial_\nu x^{\lambda_k} = \delta_\nu^{\lambda_k}$; $k = 1, ..., n-1$. Thus we can easily obtain,

$$\begin{aligned}\partial_\nu u_\mu &= a^{(2)}_{\mu\nu} + a^{(3)}_{\mu\nu\lambda_2} x^{\lambda_2} + a^{(3)}_{\mu\lambda_1\nu} x^{\lambda_1} + \cdots \\ &\quad + a^{(n)}_{\mu\nu\lambda_2\lambda_3\cdots\lambda_{n-1}} x^{\lambda_2} x^{\lambda_3} \cdots x^{\lambda_{n-1}} + \cdots \\ &\quad + a^{(n)}_{\mu\lambda_1\lambda_2\cdots\lambda_{n-2}\,\nu} x^{\lambda_1} x^{\lambda_2} \cdots x^{\lambda_{n-2}} + \cdots \quad (14)\end{aligned}$$

Since the summation rule holds for the terms containing doubly repeated indices, a subscript and a superscript, and $\lambda_1, \lambda_2, ... = 0, 1, 2, 3$, then we can write,

$$a^{(3)}_{\mu\nu\lambda_2} x^{\lambda_2} + \underbrace{a^{(3)}_{\mu\lambda_1\nu}}_{=a^{(3)}_{\mu\nu\lambda_1}} x^{\lambda_1} = 2a^{(3)}_{\mu\nu\lambda_1} x^{\lambda_1} \ .$$

One has likewise expressions for the other terms. Thus Eq.(14) can be written as,

$$\partial_\nu u_\mu = a^{(2)}_{\mu\nu} + 2a^{(3)}_{\mu\nu\lambda_1} x^{\lambda_1} + \cdots$$
$$+(n-1) a^{(n)}_{\mu\nu\lambda_1\lambda_2\cdots\lambda_{n-2}} x^{\lambda_1} x^{\lambda_2} \cdots x^{\lambda_{n-2}} + \cdots \ . \quad (15)$$

Following the step producing Eqs.(12) and (15), we can get similar expressions for $\partial_\mu x_\nu$ and $\partial_\rho x^\rho$. The last term of Eq.(11) can be found by substituting u^ρ with $g^{\rho\sigma} u_\sigma$. Thus, in its series form, the differential equation (11) can be written as,

$$\begin{aligned}&\left\{ a^{(2)}_{\mu\nu} + a^{(2)}_{\nu\mu} - \frac{1}{2} g_{\mu\nu} a^{(2)\rho}_{\rho} \right\} \\ &+2 \left\{ a^{(3)}_{\mu\nu\lambda_1} + a^{(3)}_{\nu\mu\lambda_1} - \frac{1}{2} g_{\mu\nu} a^{(3)\rho}_{\rho\lambda_1} \right\} x^{\lambda_1} + \cdots \\ &+(n-1) \left\{ a^{(n)}_{\mu\nu\lambda_1\lambda_2\cdots\lambda_{n-2}} + a^{(n)}_{\nu\mu\lambda_1\lambda_2\cdots\lambda_{n-2}} \right. \\ &\left. -\frac{1}{2} g_{\mu\nu} a^{(n)\rho}_{\rho\lambda_1\lambda_2\cdots\lambda_{n-2}} \right\} \\ &x^{\lambda_1} x^{\lambda_2} \cdots x^{\lambda_{n-2}} + \cdots = 0 \ , \quad (16)\end{aligned}$$

in which we have contracted the coefficient $a^{(n)}_{\sigma\rho\lambda_1\cdots\lambda_{n-2}}$ with $g^{\rho\sigma}$ to produce $a^{(n)\rho}_{\rho\lambda_1\cdots\lambda_{n-2}}$ with $n = 2, ..., \infty$.

Since x^{λ_k} is arbitrary and the coefficient inside $\{...\}$ is symmetrical with respect to an interchange of any pairs of indices among $\{\lambda_1, \cdots, \lambda_{n-2}\}$, then the coefficient of Eq.(16) can be extracted trivially by making each term with various power of x equal to zero, of course for x not zero. This procedure will give a set of equations which make all the coefficients in each term vanish. In other words, the homogeneous polynomials having different degrees of x give the following separate conditions for each $a^{(n)}$,

$$a^{(n)}_{\mu\nu\lambda_1\cdots\lambda_{n-2}} + a^{(n)}_{\nu\mu\lambda_1\cdots\lambda_{n-2}} = \frac{1}{2} g_{\mu\nu} a^{(n)\rho}_{\rho\lambda_1\cdots\lambda_{n-2}} \ , \quad (17)$$

with $a^{(n)}_{\mu\nu\lambda_1\cdots\lambda_{n-2}}$ and $a^{(n)\rho}_{\rho\lambda_1\cdots\lambda_{n-2}}$ totally symmetric in the $n-2$ last indices. Since Eq.(16) does not involve $a^{(1)}_\mu$, these constants can be cosen arbitrarily. For $n = 2$, it is more convenient to write back the $a^{(2)\rho}_{\rho}$ etc. to their earlier definition, *i.e.* as $g^{\rho\sigma} a_{\sigma\rho}$, which if expanded becomes,

$$\begin{aligned}g^{\rho\sigma} a^{(2)}_{\sigma\rho} &= g^{00} a^{(2)}_{00} + g^{11} a^{(2)}_{11} + g^{22} a^{(2)}_{22} + g^{33} a^{(2)}_{33} \\ &= -a^{(2)}_{00} + a^{(2)}_{11} + a^{(2)}_{22} + a^{(2)}_{33} = 4\mathcal{D} \ , \quad (18)\end{aligned}$$

where $\mathcal{D}$ is a constant which is defined by

$$-a^{(2)}_{00} = a^{(2)}_{11} = a^{(2)}_{22} = a^{(2)}_{33} = \mathcal{D} \ .$$

Hence the condition (17) for $n = 2$ can be written as

$$a^{(2)}_{\mu\nu} + a^{(2)}_{\nu\mu} = 2 g_{\mu\nu} \mathcal{D} \ . \quad (19)$$

From this equation we can conclude that the $a^{(2)}_{\mu\nu}$ coeficient can be written as

$$a^{(2)}_{\mu\nu} = \omega_{\mu\nu} + \mathcal{D} g_{\mu\nu} \ , \quad (20)$$

with $\omega_{\mu\nu}$ anti-symmetric under the exchange of its pair of indices, *i.e.* $\omega_{\mu\nu} = -\omega_{\nu\mu}$. In the same way for $n = 3$ we can define $a^{(3)\rho}_{\rho\lambda_1} \equiv 4c_{\lambda_1}$ which appears in Eq.(17), so that one gets,

$$a^{(3)}_{\mu\nu\lambda_1} = g_{\mu\nu} c_{\lambda_1} + g_{\mu\lambda_1} c_\nu - g_{\nu\lambda_1} c_\mu \ , \quad (21)$$

in order to satisfy the condition (17) for $n = 3$. The nontrivial solution of $n > 3$ cannot be found if the dimensionality of the space-time is greater than 2. This follows from the permutation symmetry of the last $n-1$ indices of $a^{(n)}$. Hence for $n > 3$ all $a^{(n)} = 0$.

Collecting the results above, the following form of u_μ emerges,

$$\begin{aligned}u_\mu(x) &= a_\mu + (\omega_{\mu\nu} + \mathcal{D} g_{\mu\nu}) x^\nu \\ &\quad + (g_{\mu\nu} c_{\lambda_1} + g_{\mu\lambda_1} c_\nu - g_{\nu\lambda_1} c_\mu) x^\nu x^{\lambda_1} \quad (22) \\ &= a_\mu + \omega_{\mu\nu} x^\nu + \mathcal{D} x_\mu + 2x_\mu c_\nu x^\nu - c_\mu x_\nu x^\nu \ ,\end{aligned}$$

or since $g^{\mu\alpha} u_\mu = u^\alpha$, it can be written also as,

$$u^\alpha(x) = a^\alpha + \omega^\alpha_\nu x^\nu + \mathcal{D} x^\alpha + 2x^\alpha c_\nu x^\nu - c^\alpha x_\nu x^\nu \ , \quad (23)$$

where a^α, $\mathcal{D}$, c_ν, c^α, and $\omega^\alpha_\nu \equiv g^{\mu\alpha} \omega_{\mu\nu}$ are arbitrary constant parameters.

Properly considered, the family of the differential operators generating the transformations (4) with u^α given by Eq.(23), $u^\alpha(x)\partial_\alpha$, are the generators of a 15-parameter Lie group, manifested by the conformal transformation group in a 4-dimensional space-time preserving the causal structure given by Eq.(1). The part with the parameters a^α (4 parameters) and $\omega_{\mu\nu}$ (6 parameters) generates the

Poincaré transformations, the part with the one parameter $\mathcal{D}$ generates dilatations. The part with the parameters c_λ (4 parameters) generates the "proper conformal transformations" (also called "conformal translations"), *i.e.* [11],

$$x'^\mu = \frac{x^\mu - c^\mu x_\mu x^\mu}{1 - 2c_\mu x^\mu + c_\mu c^\mu x_\mu x^\mu} \; . \tag{24}$$

The transformations given by Eq.(24) become singular on the sub-manifold where the denominator vanishes. Therefore they do not give global symmetries of the $\mathcal{M}$ manifold but can be defined physically only as local diffeomorfisms in suitable regions for a limited range of the parameters c_λ. One can, however, compactify $\mathcal{M}$ so that conformal transformations act as diffeomorfisms defined everywhere in the resulting space. Since Eq.(24) is a nonlinear transformation, the 15-parameter conformal group is realized in a nonlinear manner as the transformation group in the Minkowski space-time even though its Poincaré subgroup represents strictly linear transformations [12].

The problem arising from nonlinearity can be handled by using the group representations of linear transformations accommodated in 6-dimensional real space. The use of this space is based on the property of isomorfism existing between the conformal group and SO(4,2), which is the isometry group of flat space equipped with a metric [3],

$$g_{ab} = (-1)^{\delta_{\overline{a}0} + \delta_{\overline{b}5}} \delta_{\overline{a}\overline{b}} = \left\{ \begin{array}{lll} -\delta_{ab} & \text{for} & \alpha = 0,5 \\ +\delta_{ab} & \text{for} & \alpha = 1,2,3,4 \end{array} \right. , \tag{25}$$

that leaves Minkowski space-time, with a metric (3), conformally invariant. In other words, we can use a 6-dimensional manifold as a scale to support the space-time structure for describing the physical systems, two additional coordinates is shown as the scale representative and the scale exchange from a point to another point. These coordinates can be represented by ξ^a, $a = 0, 1, ..., 5$ (instead of $\mu, \nu, ...$ (greek alphabet) indices assigned as $0, 1, 2, 3, \cdots$ the $a, b, \cdots$, each spans $0, \cdots, 5$). The quadratic form is given by,

$$\xi^a \xi_a = g_{ab} \xi^a \xi^b = \xi^\mu \xi_\mu + (\xi^4)^2 - (\xi^5)^2 \; . \tag{26}$$

Instead of the coordinates ξ^4 and ξ^5, we can use new coordinates κ and η by introducing them as,

$$\kappa = \xi^4 - \xi^5 \quad \text{and} \quad \eta = \xi^4 + \xi^5 \; . \tag{27}$$

In this way Eq.(26) can be written as,

$$\xi^a \xi_a = \xi^\mu \xi_\mu + \kappa\eta \; . \tag{28}$$

Further, the pseudo-orthogonal transformation group SO(4,2), with the above requirements can be written as,

$$\xi'^a = M^a_b \xi^b \; , \tag{29}$$

with the 6×6 M matrix satisfying,

$$g_{ab} M^a_c M^b_d = g_{cd}; \qquad \det M = 1 \; . \tag{30}$$

The $M \in SO(4,2)$ transforms the light-cone into itself, *i.e.*,

$$\xi^a \xi_a = 0 \; . \tag{31}$$

Therefore, for the case governed by Eq.(31), Eq.(28) reduces to,

$$\xi^\mu \xi_\mu = -\kappa\eta \; . \tag{32}$$

Further, with the definition,

$$\xi^\mu = \kappa x^\mu \; , \tag{33}$$

one obtains ,

$$x^\mu x_\mu = -\frac{\eta}{\kappa} \; . \tag{34}$$

From the steps described above, we can determine many subgroups of the conformal group, each representing a transformation restricted to one coordinate or more and keep the other coordinates unchanged. In this way we can extract:

i) The 6-parameter subgroup obtained by rotating ξ^μ and leaving the coordinates ξ^4 and ξ^5 unchanged. It is generally well known that these rotations will yield the homogeneous Lorentz transformations. In another word, we can write,

$$\xi' = \underbrace{\begin{pmatrix} \Lambda^{\mu\nu} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{M^a_b} \xi, \qquad \Lambda \in \mathcal{L} \; . \tag{35}$$

ii) The 1-parameter subgroup obtained by rotating ξ^4 and ξ^5 and leaving the coordinates ξ^μ unchanged. These transformations leave $(\xi^4)^2 - (\xi^5)^2$ invariant and have the forms,

$$\begin{aligned} \xi'^4 &= \xi^4 \cosh\theta + \xi^5 \sinh\theta \; , \\ \xi'^5 &= \xi^4 \sinh\theta + \xi^5 \cosh\theta \; , \end{aligned} \tag{36}$$

where θ is a pseudo-rotation parameter. Using Eq.(36) we can obtain the transformation of the coordinates appearing in Eq.(27),

$$\begin{aligned} \kappa' &= \xi'^4 - \xi'^5 = e^{-\theta}(\xi^4 - \xi^5) = \mathcal{D}^{-1}\kappa \; , \\ \eta' &= \mathcal{D}\eta \; , \end{aligned} \tag{37}$$

where $\mathcal{D} = e^\theta$. Both these equations are the dilatation of κ and the inverse dilatation of η. Since under the transformation (36), ξ^μ are unchanged, then using (33),

$$\xi'^\mu = \xi^\mu = \kappa x^\mu \; ,$$

will lead to,

$$\xi'^{\mu} = \kappa' x'^{\mu} = \mathcal{D}^{-1}\kappa x'^{\mu} \ .$$

From the two equations above one gets,

$$x'^{\mu} = \mathcal{D}x^{\mu} \ , \tag{38}$$

which is the dilatation of the x^{μ}.

iii) In the same way, we can get the 4-parameter subgroup that results in translations, related to the transformation of coordinates ξ'^{μ} where κ is unchanged:

$$\begin{aligned} \xi'^{\mu} &= \xi^{\mu} + \kappa a^{\mu} \ , \\ \kappa' &= \kappa \ . \end{aligned} \tag{39}$$

Using Eq.(34) one can write $\kappa'\eta' = -\xi'^{\mu}\xi'_{\mu}$, and substitution of Eq.(39) into this relation leads to,

$$\eta' = \eta - 2\xi^{\mu}a_{\mu} - \kappa a^{\mu}a_{\mu} \ . \tag{40}$$

From Eq.(33) it is clear that $\xi'^{\mu} = \kappa' x'^{\mu}$, so that from Eq.(39) we get $\kappa x'^{\mu} = \kappa x^{\mu} + \kappa a^{\mu}$ or,

$$x'^{\mu} = x^{\mu} + a^{\mu} \ , \tag{41}$$

which is the Poincaré translation with four parameters a^{μ}.

iv) The 4-parameter subgroup that results in a conformal translation can be found by applying the coordinate transformation ξ'^{μ} with η unchanged:

$$\begin{aligned} \xi'^{\mu} &= \xi^{\mu} + \eta c^{\mu} \ , \\ \eta' &= \eta \ , \\ \kappa' &= \kappa - 2\xi^{\mu}c_{\mu} - \eta c^{\mu}c_{\mu} \ . \end{aligned} \tag{42}$$

The third of Eq.(43) is found in the same way as Eq.(40). Using the same procedure as used in step iii) one can find,

$$x'^{\mu} = \frac{\xi^{\mu} + \eta c^{\mu}}{\kappa - 2\xi^{\mu}c_{\mu} - \eta c^{\mu}c_{\mu}} \ . \tag{43}$$

We can get back Eq.(24) by substituting Eqs.(33) and (34) into Eq.(43) .

3 CONCLUSION

The 15 parameters as elements of the conformal group can be derived readily, easily and pedagogically from the displacement field of an infinitesimal transformation in the Minkowski space-time. This derivation can be carried out by using the transformation formulas connecting two coordinate systems which involves the light-cone characterized by $\Delta x = 0$ and preserve the causal structure. These 15 parameters incorporate 6 parameters which are elements of the homogeneous Lorentz group, 4 parameters of the translations, 1 parameter of the dilatations, and 4 parameters of the conformal translations.

Although this group can be defined physically only as local diffeomorfisms in suitable regions for a limited range of the parameters c_{λ}, its manifold, however, can be compactified such that these transformations act as diffeomorphisms defined everywhere in the resulting space.

The conformal group is realized in a nonlinear manner as a group of transformations in the Minkowski space. Due to the presence of isomorphism of the conformal group with the group SO(4,2), it is possible to use the group representations with linear transformations accommodated in a 6-dimensional real space.

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