DEVELOPMENT OF PUPILS' RELATIVE THINKING IN SOLVING PROPORTIONAL REASONING PROBLEMS THROUGH COMPARISON TASKS

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Abstract

The concept of proportions is not only fundamental to educational topics, but it is also essential for everyday competence. However, proportional reasoning is difficult for many pupils, even it is not easy for adults. Pupils often find it hard to determine an appropriate reasoning to be used in a certain situation. One common error that many pupils should encounter is the misuse of absolute comparison in situation requiring relative comparison. Many pupils compare the situations in partial way without recognizing relative relationship between data. They tend to compare the absolute value instead of considering the proportional relationship of data. In order to support the development of pupils' relative thinking in solving proportional reasoning problems, we designed a series of learning activities involving comparison tasks. Moreover, the present study is aimed to contribute to developing a local instruction theory about the domain. Consequently, design research is chosen as an approach to achieve the research aims. In addition, we designed the teaching-learning activities based on the tenets of Pendidikan Matematika Realistik Indonesia (PMRI) that was adapted from Realistic Mathematics Education (RME). The paper focuses on the first cycle of the study involving nine pupils of 5th grade elementary school. The findings on learning experiment shows that comparison tasks may promote pupils' relative thinking in solving proportional reasoning problems.

Keywords: relative thinking, proportional reasoning, comparison tasks, RME, design research

INTRODUCTION
The concept of ratios and proportions are essential to success in many educational topics (Lesh, Post, & Behr 1988; Lamon, 2007; Ben-Chaim, Keret, &, Ilany, 2012). Furthermore, there are many familiar situations in daily life that involve the concept of proportion, i.e.: in enlarging and reducing photos and copies, price comparisons (van Galen, Feijs, Figueiredo, Gravemeijer, Herpen, &Keijzer, 2008), density, speed and ingredients of a recipe (Karplus, Pulos, & Stages, 1983).
According to Bright, Joyner, & Wallis (2003), proportional reasoning needs relational thinking including the use of ratios in comparison quantities. In line with that, Hilton, Hilton, Dole, & Goose (2013) claimed that an ability to distinguish relative and absolute situation is important in proportional reasoning. The above findings (Bright, *et.al*, 2003; Hilton, *et.al*, 2013) justified Lamon's (2006) statement that an ability to analyze situation in absolute and relative perspective is one of the most important types of thinking required for proportional reasoning.
Sumarto, van Galen, Zulkardi, & Darmawijoyo (2014) conducted a study in assessing pupils' proportional reasoning involving relative thinking problems. In the study, pupils

were asked to compare situations. Based on result of the study, Sumarto, *et al.*, (2014) concluded that in order to solve comparison problems, one needs to think about the relationship of numbers (data) and compute the set number for each situation in order to determine which proportion that gives a good comparison. It implies that working on comparison tasks might help pupils to elicit the relative thinking that is important in proportional reasoning. Furthermore, Lesh, Post, & Behr as cited by Hilton, *et al.*, 2013 stated that it is important to provide a range of situations in which pupils have more opportunity to employ proportional reasoning. Therefore, the present study is aimed to develops instructional activities about the domain involving varies proportional situations to foster pupils' relative thinking.

However, most studies mentioned above (Bright, *et al.*, 2003; Hilton, *et al.*, 2013) deal more with the theory and contribute less to the practice of teaching. Therefore, it is needed to conduct a study that contribute for the development of theory and can be applied as teaching instructions. Due of that, the present study is aimed to develop instructional activities that can be applied in classroom and contribute to developing local instruction theory that supports pupils to elicit their relative thinking in solving proportional problems. Hence, in order to achieve the goals, design research is chosen as an approach in this study. In this study we will develop a learning trajectory and instructional activities so as to support the development of pupils' relative thinking in solving proportional reasoning.

The present study is part of a big study that involve two cycles, namely pilot teaching experiment (first cycle) and teaching experiment (second cycle). This article presents and discusses the data from the first cycle. The focused topic of this paper is investigating how comparison tasks can support the development of pupils' relative thinking in solving proportional reasoning problems. Thus, we address a question: *how can comparison tasks promote pupils' relative thinking in solving proportional reasoning problems?*

THEORETICAL FRAMEWORK
Proportion

Tourniaire & Pulos (1985) defined a proportion is a statement of equality of two ratios, i.e., $\frac{a}{b} = \frac{c}{d}$. Langrall & Swafford (2000) justified that a proportion is a statement in which two ratios are equal in the sense that both express the same relationship. Therefore, it is clear that a proportion denotes a relationship between numbers. Therefore, in solving proportional reasoning problems, one should consider the relationship of numbers instead of looking upon the absolute value.

Generally, there are two types of proportional reasoning problems, missing value problems and comparison problems (Karplus, *et al.*, 1983; Tourniaire & Pulos, 1985; Silvestre &da Ponte, 2012). Tourniaire & Pulos (1985) explained that a missing value problem is usually presented with three numbers, *a, b,* and *c,* and the task is to find the unknown *x* such that $\frac{a}{b} = \frac{c}{x}$. Meanwhile, in comparison problems, a pupil should compare two values of the intensive variable computed from the data (Karplus, *et al.*, 1983). one compares two or more situations, he/she should consider the relation between numbers (data) and computes the set of numbers for each situation in order to determine which proportion gives a good comparison (Sumarto *et al.*, 2014). Align with that, in solving proportional reasoning problems, a pupil should analyze the situation in different perspective, absolute and relative perspective.

Relative thinking

Lamon (2006) explains that when someone see a situation in relative perspective, he/she considers the relationship of an actual number or data to the other number or data, for example: if a child analyzes the growth of a tree in relative way, he/she will think about the relationship of the actual growth with the initial height, whereas if the child considers the change in an absolute way, he/she will count the difference of the present height and the initial height (the actual growth) only. The notion of relative thinking is in accordance with how people should solve a comparison problem (Sumarto *et al.*, 2014).

RME

The learning instructions are designed based on tenets of *Pendidikan Matematika Realistik Indonesia*. This approach was adapted from Realistic Mathematics Education.

- **The use of context**

  In this study, we design contextual problems as the mathematics activity. The contextual problems are derived from daily life contexts which are familiar for pupils. The use of contextual problems makes the mathematics becomes meaningful for pupils.

- **The use of model**

  Models are representation of pupils' thinking and it is concrete for pupils. It is easier for pupils to work and reason in something which is concrete for them. And then, because the models represent the initial context situations, it is easy for pupils to relate their works and reasoning by using models and the initial problems.

- **The use of pupils' contribution**

  In a learning process, teacher should not be the one who judges the acceptable answer. It is important to give a wide opportunity for pupils to participate and contribute actively, because the development of individual's reasoning can't be separated from his/her participation in sharing mathematical meaning (Cobb & Yackel, 1996). Therefore, the instructional activities are design to encourage pupils to contribute actively in the learning process.

- **Interactivity**

  Van den Heuvel-Panhuizen, (2001) claims that learning of mathematics is a social process. Because of that, the tasks are designed as working group activities. Moreover, the teacher also interacts with the pupils during the working group. Besides that, a whole-class discussion is a chance for every member of the class to interact each other.

- **Intertwining**

  Proportional reasoning involves the use of ratios in comparing quantities (Bright, Joyner, & Wallis, 2003). Therefore, using fractions as mathematics tools to solving the comparison problems proportionally is one of learning goals.

Proportion in Indonesian curriculum

According to NCTM Curriculum and Evaluation Standard (1989), proportional reasoning develops in pupils throughout grade 5-8. In the Indonesian curriculum, the concept proportion is formally taught start at grade 5. Moreover, at grade 4 pupils have learnt about fraction and solving problems involving fractions that are also related to proportional reasoning. Therefore, it provides an opportunity to employ proportional problems in a more advance level at grade 5. Hence, this study will be conducted with 5th

graders and their teacher in Indonesia. The following table presents standard competence and basic competence used in this study.

Table 1. Proportion for elementary school in the Indonesian curriculum (Depdiknas, 2006)

Standard Competence	Basic Competence
Numbers **2.** Using fractions to solve problems	5.1 Converting fractions into decimals and vice versa.
	5.2 Adding and subtracting fractions and decimals.
	5.3 Multiplying and dividing fractions and decimals.
	5.4 Using fractions in solving proportional problems and scale

METHOD

Research method

In this study, we design four different learning activities that are integrated and support each other. The first activity supports the second activity; the second activity supports the third activity; and so on. Each learning activity is designed to achieve particular learning goals. Furthermore, the study is aimed to contribute to developing a local instruction theory about this domain and to get comprehensive understanding about how the designed instructional activity works. Due of that, design research is chosen as an approach to achieve the research goals.

Participants

In the first cycle, the participants are nine pupils of grade 5 SD Negeri 1 Palembang, Indonesia. The pupils were chosen by considering their achievement at studying. However, the pupils weren't not extremely clever or extremely low achiever.

Data collection and method analysis

The aim of the present study is to get comprehensive description about how comparison tasks support pupils to elicit relative thinking in solving proportional reasoning problems. Therefore, data collection will be analyzed qualitatively.

A week before doing the learning activity, the pupils worked on a pre-test. There are three problems in the pre-test that are aimed to determine pupils' preliminary knowledge. The three problems are comparison problems that are used to identify whether pupils solve the problems by using the concept of proportionality or compare the absolute value. In the end of lesson, the pupils did post-test. The post-test was carry out in order to know how the instructional activities (learning activities and supporting from teacher) supported the development of pupils' relative thinking in solving proportional reasoning problems.

There are four learning activities in the design. This paper focuses on two learning activities, which each of them is implemented in different lesson. The learning process is video recorded. The data collections of the first cycle are pupils' written works, video registered, pupils' interview and field notes. In analyzing the data collection, we used the initial hypothetical learning trajectory (HLT) as a guideline. The data from the learning process were analyzed against the initial HLT. The researcher then used the result of analyzing data as consideration in improving the initial HLT. The revised HLT then is implemented in the second cycle that is a real teaching experiment in an actual learning environment.

RESULT AND DISCUSSION

The article focuses on two learning activities that will be elaborated in the following section.

Comparing density

Pupils' task in learning activity 1 is comparing situation of four chicken boxes. The pupils should determine which chicken box is the most crowded. Besides that, they are asked to put the boxes in an order, from the most crowded to the least crowded. Before pupils worked on the main activity, the teacher gave preliminary activity in which pupils did mind experiment as follows:

Which space is more crowded, A or B, both space are occupied by 10 objects, but B is twice larger than space A?

The following fragment shows pupils' answer and reasoning:

| 1 | Pupil 1 | : | Space A is more crowded, because it is smaller than B and both of them are occupied by the same number of object. So, B is more spacious. |

In order to expand pupils' thinking, the teacher asked this follow up question:

2	Teacher	:	There are two desks, desk A and B. One pupil occupies desk A and two pupils occupy desk B. Which one is more crowded?
3	Pupil 2	:	Desk B is more crowded, because both desks equal in size but there are more pupils in desk B.
4	Teacher	:	So, in this case, what's your consideration in determining which desk is more crowded?
5	Pupil 1	:	The number of pupils.

At line 1 and 5 in the above fragment, we could see that the pupil compared the situation in absolute way. The pupils didn't consider the relationship between the number of objects and the size of space. At line 1, the pupil compared the absolute size of space, and they explained that smaller space is more crowded. At line 4 and 5, the pupil compared the number of objects, and they came up with the idea that more pupils occupied a space, then the space be more crowded. But, at line 3, Pupil 2 also noticed that since the two desks are equal in size, they could just compare the number of pupils. It indicates that Pupil 2 realized size of space and the number of object altogether influence the density on space.

Due of that, the teacher gave follow up questions:

| 6 | Teacher | : | What if, there are desk C and desk D, which desk D is twice larger than desk A, and there are one pupil occupies desk C, two pupils occupy desk D. Which desk is more crowded? |
| 7 | Pupil 3 | : | Those are same. Because desk D is twice larger than desk C, and there are two pupils occupy desk D. Therefore, desk D is divided for two pupils. That's way, desk C and D equal in density. |

Based on line 7, it is clear that Pupil 3 considered the relationship between size of the space and number of objects. Pupil 3 saw the situation in relative way instead of comparing the absolute value. Moreover, Pupil 3 computed the set of numbers (size of the space and the number of objects) in each desk C and D. By referring to her calculation, Pupil 3 determined that desk C and D were equal in density. Based on pupil's

discourse above, several pupils seemed already grasping the idea of comparing and seeing situations in relative perspective.

The main activity in the first lesson contained a problem about comparing density of four chicken boxes. The four chicken boxes were different in size. Each box contained different number of chicken.
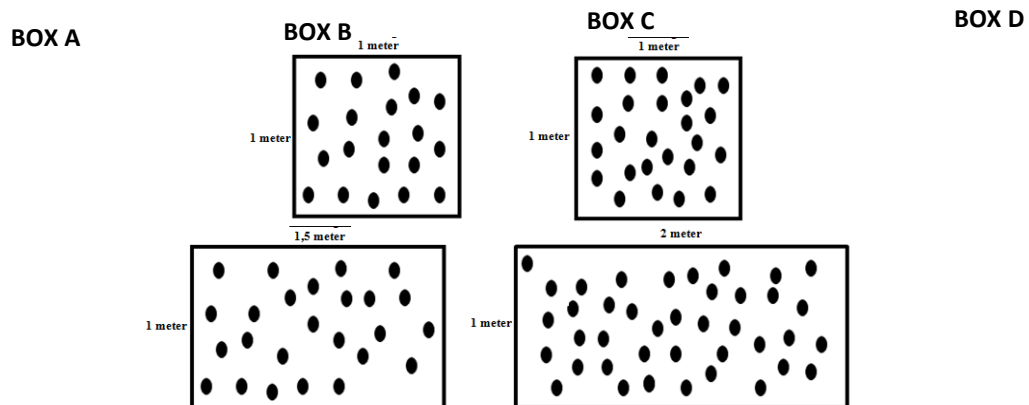

Figure 1. Chickens boxes

The pupils were asked to put the boxes in an order, from the most crowded box to the least crowded.

The preliminary activity has generated the idea of relative perspective in analyzing comparison situations. However, it was difficult for pupils bring upon the concept of proportionality in solving the proposed problem. The following figures are several pupils' answers:

Figure 2. The answer of Pupil 2 and Pupil 4

Translation of pupils' answer: box B is the most crowded, because box B is 1m² and it contains 25 chickens, and box A is 1m² that contains 20.

Figure 3. The answer of Pupil 3 and Pupil 5

Translation of pupils' answer: the order is box B, A, C and D. box A equals box B, but box B contains more chickens, so box B is more crowded. Box A and C are different in size, but box C is bigger than box A, and box B contains more chickens. Box D the biggest one and it contains most chickens.

Based on pupils' answers, we can see that the pupils got difficulty to compare density of boxes which were different in size and contained different umber of objects (chickens). At the above pupils' solutions, both groups compared the density of boxes that had equal size (box A and B). Because the size of box A equal box B, and box B contained more chicken, so the pupils concluded that box B was more crowded.

In comparing the density of boxes which were different in size and contained different umber of objects (chickens), it wasn't clear how and why the pupils did come up with the solution. It indicates that pupils didn't understand how they should solve the problems. It implies that the pupils needs more support. The support can be scaffolding from teacher of strong follow up question. In order to make scaffolding and follow up questions be more effective, the teacher should use right words.

In comparing-density activities, the pupils have already had experiences about using the concept of proportionality instead of absolute value to solve proportional problems. That starting point is necessary for working at next learning activity. In order to expand pupils' reasoning about proportional-comparison task, we designed different context for each learning activity. By experiencing proportionality in different contexts, it might foster pupils' proportional reasoning.

In the next section, the article focuses on learning activity about using data of survey to derive a good comparison.

Survey on pupils' interest

The goals of "**Survey on Pupils' Interest**" activity is supporting pupils to solve problems in relative on proportional comparison in term of part to whole and the pupils would be able to use fraction as tools to solve the problems. In this third learning activity, the pupils were given data of survey on children's interest, and then they were asked to compare situations by using the given data. To solve the problems, the pupils might use absolute thinking or they might apply relative comparison in term of part-to-whole.

The first problem was about children' (boys and girls) interest in traditional dance. The pupils were asked to determine who, boys or girls, were more interested in traditional dance.

Table 2. Children preference on traditional dance

Girls

Girls who like to do traditional dance	Girls who don't like to do traditional dance	Total girl pupils
100	150	250

Boys

Girls who like to do traditional dance	Boys who don't like to do traditional dance	Total boys pupils
56	94	150

In solving the first problem, all pupils stated that the number of girls who liked to do traditional dance was more than boys (100 > 56). Therefore, they concluded that girls were more interested on traditional dance than boys.

The aim of giving the first problem, which the pupils could use both absolute and relative perspective, is to help pupils understanding comparison situation in different perspective. The idea was not that one perspective is wrong and the other was correct. Both perspectives were useful and it depended on the situations. Therefore, in the second problems, it was designed a problem that pupils should see the situation in different way instead of seeing the situation in absolute way.

The second problem was about children' (boys and girls) interest in *Basketball* extracurricular. The pupils were asked to determine who (boys or girls) were more interested in *Basketball* extracurricular.

Table 3. Children preference on basketball activity

Girls

Girls who like to do basketball activity	Girls who don't like to do basketball activity	Total girl pupils
100	250	250

Boys

Boys who like to do basketball activity	Boys who don't like to do basketball activity	Total boys pupils
75	75	150

In the hypothetical learning trajectory (HLT), the researcher made two kinds of conjectures of pupils' responses. The pupils might compare the absolute number of children (boys/girls) who liked to do basketball activity, or they might compare the absolute value of total girls and boys. On the other way, the pupils might employ the concept of proportion (part relative to the whole).

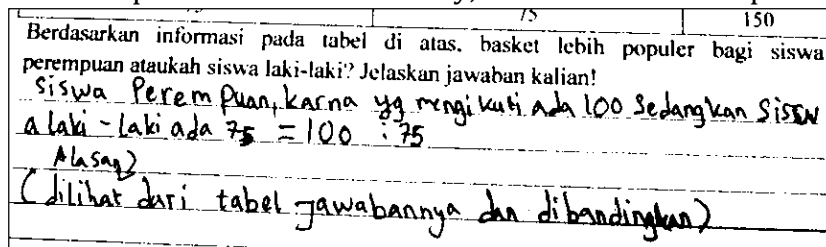Most pupils solved the problem in absolute way, one of them was Pupil 1's answer:

Figure 4. Pupil 1's answer for problem 2

Translation of pupils' answer: Girls are more interested in basketball activity. Because there are 100 girls like to do it and there are just 75 boys like to do it. So, 100:75.

In a class discussion, the teacher asked pupils to elaborate their answers and the reasoning.

8	Pupil 1	:	Because, 100 girls like to do it (*she meant that there were 100 girls who like to do basketball activity*).
9	Pupil 1	:	Nah, there are just 75 boys who like to do it (she meant that there were 75 boys who like to do basket ball activity)
10	Teacher	:	There are 75 boys who like to do it. So in this case, what do you compare?
11	Pupil 2	:	100 and 75

Both class discussion and pupils' answer showed that pupils compared the absolute number of girls (100) and boys (75) who liked to do basketball activity.

One pupil, Pupil 3, gave a different answer. But it wasn't clear what she meant because she answered that there were just 150 boys.

Berdasarkan informasi pada tabel di atas, basket lebih populer bagi siswa perempuan ataukah siswa laki-laki? Jelaskan jawaban kalian!
Laki Laki _Karena_ _Siswa_ _Laki-Laki_ _hanya._
ada _150 Siswa_

Figure 5. Pupil 3's answer for problem 2

Then, the teacher clarified pupils' thinking and strategy in the following discourse.

12	Pupil 3	:	The answer is boys
13	Teacher	:	Why?
14	Pupil 3	:	Because, total number of boys is less than total number of girls (150 < 250)
15	Teacher	:	Wait, what do you meant by the total number of boys is less than the total number of girls?
16	Pupil 3	:	Kan, like this, eee, usually, boys are more interested on playing basketball than girls. Just because the total number of girls is bigger than the total number of boys, but, the girls who like to do it is less than a half (*she meant that the number of girls who like to it is less than a half of the total number of girls*).
17	Teacher	:	Wait, what did you say? It is less than a half? Can you repeat your previous explanation?
18	Pupil 3	:	Nah, 150 (*total number of boys*) is less a hundred than 250 (*total number of girls*), So, this is reasonable if there is less boys who like to do basket, because the total number of boys is less than girls.
19	Teacher	:	So, your answer is that boys are more interested on basket than girls?
20	Pupil 3	:	*(Pupil 3 nodded her head)* For the boys, it is a half Mam.
21	Teacher	:	How do you come up with "a half" (1/2)
22	Pupil 3	:	Because, 150 divided by 75 is 2, 2 times 75 is 150. So, it is a half (*she meant that 75 is a half of 150*)

According to above fragment we can see that Pupil 3 considered the relation between numbers (relative thinking) instead of comparing the absolute numbers of girls (100) and boys (75) who liked to do basketball activity. In line 16 it is clear that rather than of comparing the numbers of children (boys/girls) who like to do basket (part), Pupil 3 also thought about the total number of boys/girls (whole). From line 14, 16 and 18 in the above fragment, it can be seen that Pupil 3 realized the numbers of boys/girls who like to do basket and the total number of boys/girls altogether influenced the comparison situation. Moreover, line 16 and 20 showed that Pupil 3 was able to describe the part-whole relationship in each set of numbers for each situation.

In order to solve the problem, the student aimed to determine the biggest fraction among the situation. However, it was difficult for pupils because the numbers seemed too big for them. It also happened for the third problem. The pupils got difficulty in comparing the fractions.

The pupils' strategy in solving "**Survey on children interest**" activity was consistent with the researcher's prediction on the HLT. Several pupils solved in absolute way and there was a student who employed proportionality in part-whole relationship. But, the pupils had difficulties in computing the numbers. Due of that, it was necessary to change the numbers with numbers which the pupils are familiar with. For instance, by using numbers that can be imagined by pupils, or by using numbers that relates to the amount of object around the pupils.

CONCLUSIONS

According to result of the present study, we conclude that comparison task might promote pupils' relative thinking in solving proportional reasoning problems. It supports the findings of previous study by Sumarto, et al. (2014) which showed that comparison situations endorse pupils to think in absolute or relative perspective, which pupils should use relative comparison instead of absolute comparison to solve the proportional situations. The findings in this study also contribute to the developing of local instruction theory of the domain. Moreover, the findings are guideline to improve the HLT, the details of learning instruments and teacher guides that will be implemented in the teaching experiment of second cycle.

In fact of limited setting in implementing the instructional design affects the data collection obtained in this study. The design works in a learning situation that provides wide opportunity for pupils to deliver their opinion, reasoning and argumentation. Therefore, the discussion section takes essential role in the learning process. Moreover, the roles of teacher, i.e. how does teacher propose questions, what kind of question does the teacher use to scaffold the pupils, how does teacher react to pupil's particular response and how does the teacher orchestra the class discussion, are fundamental in supporting his/her pupil . Besides that, the use of number influences pupils reasoning. The use of appropriate numbers will encourage pupils to reason about the problems. However, the use of too difficult numbers will consume too much time in computing rather than in reasoning.

Based on data analysis, it is necessary to do several improvements toward the details if the learning activities, i.e. the use of numbers, and teacher guide. It is needed to choose more appropriate words in questioning. In addition, it is important to ask follow up question to clarify pupils' answers, reasoning and ideas. By using follow up question, the teacher can encourage pupils to think deeper.

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